TECHNICAL APPENDIX
Solving the Stochastic Growth Model with a Finite Element Method

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First Draft: January, 1993
Current Draft: August, 1993
Details of the calculation of approximate consumption functions for the stochastic growth model are provided below.

1. Initialization.
   
i. specify parameters of utility and production: $\beta, \delta, \tau, \rho, \alpha,$ and $\sigma_e$;
   
ii. specify the grid on capital: $\vec{k} = [k_1, k_2, k_3, \ldots, k_{n_k}]$;
   
iii. specify the grid on technology: $\vec{\theta} = [\theta_1, \theta_2, \ldots, \theta_{n_\theta}]$;
   
iv. specify an initial guess for consumption: $c(k_i, \theta_j)$ for each $k_i, \theta_j$ pair where $k_i, i = 1, \ldots n_k$, is an element of $\vec{k}$ and $\theta_j, j = 1, \ldots n_\theta$, is an element of $\vec{\theta}$; let $c_a$ denote consumption at grid point (or node) $a$ and $\vec{c}$ is the vector with elements $c_a, a = 1, \ldots, n_{nodes}, n_{nodes} = n_k n_\theta$;
   
v. specify parameters for quadrature:
      
a. for integration with respect to $\epsilon$, specify $[\epsilon, \bar{\epsilon}]$ and $m_\epsilon$;
      
b. for integration with respect to $k$ on interval $[k_i, k_{i+1}]$, specify $m_{k,i}$;
      
c. for integration with respect to $\theta$ on interval $[\theta_i, \theta_{i+1}]$, specify $m_{\theta,i}$;
   
vii. specify the maximum number of iterations of the Newton-Raphson algorithm and a criterion for stopping iterations (i.e., a norm $\| \cdot \|$ and a tolerance parameter).

2. Calculation of abscissas and weights for quadrature.
   
i. given parameters of step 1(v), calculate abscissas and weights for integration done in step 3. (See Press, et.al. (1986) for details.)

3. Main loop.
   
do $t = 1$ to maximum number of iterations
      
set $H = 0, J = 0$ where $H$ is $n_{nodes} \times 1$ and $J$ is $n_{nodes} \times n_{nodes}$
      
do $\epsilon = 1$ to number of elements ($= (n_k - 1)(n_\theta - 1)$)
         
determine current intervals for $k, \theta$ in element $\epsilon$
         
determine the 4 global nodes for element $\epsilon$ (call them nodes 1-4)
      
do $j = 1$ to number of quadrature points for $\theta$ on element $\epsilon$
         
do $i = 1$ to number of quadrature points for $k$ on element $\epsilon$
            
calculate consumption $c$ (for $(i,j)$th quadrature point in element $\epsilon$)
e.g., if \((k, \theta)\) are the current coordinates and element \(e\) is given by 
\[
[\vec{k}_e, \vec{k}_e] \times [\theta_e, \theta_e],
\]
then
\[
c = c_{\text{node}1} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \theta_e}{\theta - \theta_e} + c_{\text{node}2} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \theta_e}{\theta - \theta_e} +
\]
\[
c_{\text{node}3} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \theta_e}{\theta - \theta_e} + c_{\text{node}4} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \theta_e}{\theta - \theta_e}
\]
calculate next period capital, 
\[
K = \theta k^\alpha - c + (1 - \delta)k
\]
calculate next period’s marginal product of capital 
\[
mp_k = \alpha K^{\alpha - 1}
\]
set \(s = 0\)
do \(l = 1\) to \(m_e\)
set \(e\) equal to the \(l\)th quadrature point (calculated in step 1(v)a)
calculate the next period shock, 
\[
\Theta = \theta^\theta \exp(\varepsilon)
\]
determine the element that includes point \((K, \Theta)\) and call it \(E\)
determine the 4 global nodes for element \(E\) (denote by NODEs 1-4)
calculate next period consumption \(C\),
\[
\text{e.g., if } (K, \Theta) \text{ are the coordinates and } E \text{ is the element, then}
\]
\[
C = c_{\text{node}1} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \Theta}{\theta - \theta_e} + c_{\text{node}2} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \Theta}{\theta - \theta_e} +
\]
\[
c_{\text{node}3} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \Theta}{\theta - \theta_e} + c_{\text{node}4} \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \Theta}{\theta - \theta_e}
\]
calculate next period’s marginal utility (\(mu = \beta C^{-\gamma}\))
add \(mu \times \Theta mp_k + 1 - \delta \times f(\varepsilon) \times l\)th quadrature weight) to \(s\)
calculate derivatives of \(s\) with respect to \(c_a\) for all nodes \(a\)
end loop over \(l\)
compute residual: 
\[
R = s - c^{-\gamma}
\]
update elements of \(H\) associated with global nodes for element \(e\),
e.g., if \((k, \theta)\) are the current coordinates, if element \(e\) is given by 
\[
[\vec{k}_e, \vec{k}_e] \times [\theta_e, \theta_e],
\]
and if \(\omega_{i,j}\) are the \((i,j)\) quadrature weights, then
\[
H(\text{node 1}) = H(\text{node 1}) + \omega_{i,j} \times R \times \frac{k - k_e}{k - k_e} \cdot \frac{\theta - \theta_e}{\theta - \theta_e}
\]
\[
H(\text{node } 2) = H(\text{node } 2) + \omega_{i,j} \times R \times \frac{k - k}{k_e - k_e} \cdot \frac{\theta - \theta}{\theta_e - \theta_e}
\]

\[
H(\text{node } 3) = H(\text{node } 3) + \omega_{i,j} \times R \times \frac{k - k}{k_e - k_e} \cdot \frac{\theta - \theta}{\theta_e - \theta_e}
\]

\[
H(\text{node } 4) = H(\text{node } 4) + \omega_{i,j} \times R \times \frac{k - k}{k_e - k_e} \cdot \frac{\theta - \theta}{\theta_e - \theta_e}
\]

calculate derivatives of \(H(\text{node } l), l = 1, 2, 3, 4\) with respect to \(\theta\) unknowns \(c_a\) for all nodes \(a\) and put the answer in the node \(l\)th ...
row of \(J\) (this calculation uses the derivatives of \(s\) and \(c^{-\tau}\))
end loops over \(i, j\)
end loop over \(e\)

if \(k_1 = 0\), enforce \(c(0, \theta) = 0\), i.e., eliminate elements for \(k = 0\) nodes from \(H\) and \(J\)
solve \(Jx = H\) for \(x\) and update the consumption vector: \(\bar{c} = \bar{c} - x\)
if \(\|x\| < \text{tolerance}\), go to step 4
end loop over \(t\)

4. Print results.