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NOTES
Shocks in the CKM Economy with Convex Demand

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These notes provide derivations for a modification to the economy with convex demand in our *Econometrica* 2000 paper entitled "Sticky Price Models of the Business Cycle: Can the Contract Multiplier Solve the Persistence Problem?"

We have chosen to use capital letters for variables that are not logged and small letters for their logged values. Our notation is such that $z_{i,t} = \log\ Z(i, s^t)$ and $z_t = \log\ Z(s^t)$. Any exceptions will be noted.

1. Nomenclature

- $C(s^t)$: consumption in state s^t
- $L(s^t)$: labor supply in state s^t
- $Y(s^t)$: aggregate output in state s^t
- $Y(i, s^t)$: intermediate good of type i in state s^t
- $L(i, s^t)$: labor used by monopolist i in state s^t
- $K(i, s^t)$: capital stock of monopolist i in state s^t
- $X(i, s^t)$: investment made by monopolist i in state s^t
- $V(i, s^t)$: unit costs paid by monopolist i in state s^t
- $P(i, s^t)$: price charged by ith monopolist in state s^t
- $P(s^{t-1})$: prices currently being set, which are conditional on state s^{t-1}
- $\bar{P}(s^t)$: aggregate price level in state s^t
- $M(s^t)$: nominal money balances in state s^t

- $\mu(s^t)$: growth in nominal money balances, $t-1$ to t
- $W(s^t)$: wage rate in state s^t
- $Q(s^\tau|s^t)$: price in state s^t of a claim to one dollar in s^τ
- $U(C, L, M/\bar{P})$: utility function
- $F(K, L)$: production function
- $D(P_i/\bar{P}, \{Y_j/Y\})$: input demand functions for intermediate good i
- $\phi(X/K)$: adjustment cost function

2. Model Economy

Since we will use the first order conditions over and over again in these notes, we start with a statement of the optimization problems solved by all of the agents in the economy and the associated first order conditions.

The problem solved by the final goods producers each period is

$$(1) \qquad \max_{Y(i,s^t)} \ \bar{P}(s^t) - \int P(i, s^{t-1}) Y(i, s^t)/Y(s^t)\, di$$

subject to

$$(2) \qquad \int g\left(\frac{Y(i,s^t)}{Y(s^t)}, \lambda(s^t)\right) di = 1$$

where $\lambda(s^t)$ is an exogenous shock. The first order conditions for this problem are

$$(3) \qquad P(i, s^{t-1}) = \mu(s^t) g_1\left(\frac{Y(i,s^t)}{Y(s^t)}, \lambda(s^t)\right)$$

where μ is the Lagrange multiplier on the constraint (2). The zero-profit condition,

$$(4) \qquad \bar{P}(s^t) = \int P(i, s^{t-1}) Y(i, s^t)/Y(s^t)\, di,$$

and the first order condition for P_i imply the following for the relative price

$$(5)\qquad \frac{P(i)}{\bar{P}} = \frac{g_1\left(\frac{Y(i)}{Y},\lambda\right)}{\int g_1\left(\frac{Y(j)}{Y},\lambda\right)\frac{Y(j)}{Y}\,dj}.$$

Inverting this equation gives the input demand functions

$$(6)\qquad Y(i,s^t) = D\left(\frac{P(i,s^{t-1})}{\bar{P}(s^t)}\int g_1\left(\frac{Y(j,s^t)}{Y(s^t)},\lambda(s^t)\right)\frac{Y(j,s^t)}{Y(s^t)}\,dj,\lambda(s^t)\right)Y(s^t)$$

where $D \equiv (g_1)^{-1}$.

If we assume that $g(y,\theta) = y^\theta$, then we have:

$$(7)\qquad Y(i,s^t) = \left[\frac{\bar{P}(s^t)}{P(i,s^{t-1})}\right]^{\frac{1}{1-\theta}} Y(s^t)$$

$$(8)\qquad \bar{P}(s^t) = \left[\int P(i,s^{t-1})^{\frac{\theta}{\theta-1}}\,di\right]^{\frac{\theta-1}{\theta}}.$$

The problem solved by consumers is

$$(9)\qquad \max \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t)\, U(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t)),$$

subject to the sequence of budget constraints

$$(10)\qquad \bar{P}(s^t)C(s^t) + M(s^t) + \sum_{s_{t+1}} Q(s^{t+1}|s^t)B(s^{t+1}) \le \bar{P}(s^t)W(s^t)L(s^t) + M(s^{t-1}) + B(s^t) + \Pi(s^t) + T(s^t), \quad t = 0,1,\ldots,$$

and borrowing constraints $B(s^{t+1}) \ge \bar{B}$ for some large negative number $\bar{B}$.

The first order conditions for the consumer are therefore given by the following equations:

$$(11)\qquad -\frac{U_l(s^t)}{U_c(s^t)} = W(s^t)$$

$$(12)\qquad \frac{U_c(s^t)}{\bar{P}(s^t)} = \beta\sum_{s_{t+1}} \pi(s^{t+1}|s^t)\frac{U_c(s^{t+1})}{\bar{P}(s^{t+1})} + \frac{U_m(s^t)}{\bar{P}(s^t)}$$

$$(13)\qquad Q(s^\tau|s^t) = \beta^{\tau-t}\pi(s^\tau|s^t)\frac{U_c(s^\tau)}{U_c(s^t)}\frac{\bar{P}(s^t)}{\bar{P}(s^\tau)} \quad \text{for all } \tau > \text{t}$$

where $U(s^t)$ is shorthand for $U(C(s^t), L(s^t), M(s^t)/\bar{P}(s^t))$.

The problem solved by the monopolist adjusting his price is to choose $P(i, s^{t-1})$, $K(i, s^\tau)$, $X(i, s^\tau)$, and $L(i, s^\tau)$ $\tau = t, \ldots, t+N-1$ to maximize

$$(14) \quad \sum_{\tau=t}^{\infty} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \left[P(i, s^{t-1}) Y(i, s^\tau) - \bar{P}(s^\tau) W(s^\tau) L(i, s^\tau) - \bar{P}(s^\tau) X(i, s^\tau) \right]$$

subject to the demand for good i in (6), the production technology:

$$(15) \quad Y(i, s^t) = F(K(i, s^{t-1}), L(i, s^t))$$

and the law of motion for capital used in producing i:

$$(16) \quad K(i, s^t) = (1-\delta) K(i, s^{t-1}) + X(i, s^t) - \phi\left(\frac{X(i, s^t)}{K(i, s^{t-1})}\right) K(i, s^{t-1}).$$

The first order conditions for the case with $F(K, L) = K^{\alpha_1} L^{\alpha_2}$ are given by

$$(17) \quad \sum_{\tau} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \Big\{ Y(i, s^\tau) + Y(s^\tau) \left[1 - \bar{P}(s^\tau) V(i, s^\tau) / P(i, s^{t-1}) \right] D_1 \left(\frac{P(i, s^{t-1})}{\bar{P}(s^\tau)} \int g_1 \left(\frac{Y(j, s^\tau)}{Y(s^\tau)}, \lambda(s^\tau) \right) \frac{Y(j, s^\tau)}{Y(s^\tau)} \, dj, \lambda(s^\tau) \right) g_1 \left(\frac{Y(i, s^\tau)}{Y(s^\tau)}, \lambda(s^\tau) \right) \Big\} = 0$$

$$(18) \quad V(i, s^t) = W(s^t) / F_l(i, s^t)$$

$$(19) \quad \frac{1}{1 - \phi'(i, s^t)} = \sum_{s_{t+1}} \frac{Q(s^{t+1} | s^{t-1}) \bar{P}(s^{t+1})}{Q(s^t | s^{t-1}) \bar{P}(s^t)} \Big[V(i, s^{t+1}) F_k(i, s^{t+1}) + \frac{1}{1 - \phi'(i, s^{t+1})} \left\{ 1 - \delta - \phi(i, s^{t+1}) + \phi'(i, s^{t+1}) \frac{X(i, s^{t+1})}{K(i, s^t)} \right\} \Big]$$

where $F(i, s^t)$ and $\phi(i, s^t)$ are shorthand for $F(K(i, s^{t-1}), L(i, s^t))$ and $\phi(X(i, s^t)/K(i, s^{t-1}))$, respectively. The monopolists not setting prices will still maximize with respect to labor, investment, and capital. Therefore, there will be one pricing equation and N Euler equations for capital. The first order conditions for those monopolists not setting prices depend on the prices that they last set.

Finally, the following equilibrium constraints must hold:

$$
\begin{aligned}
(20) \quad M(s^t) &= \mu(s^t)M(s^{t-1}) \\
(21) \quad T(s^t) &= M(s^t) - M(s^{t-1}) \\
(22) \quad L(s^t) &= \int L(i, s^t)\, di \\
(23) \quad Y(s^t) &= C(s^t) + \int X(i, s^t)\, di.
\end{aligned}
$$

To summarize, we have equations (4)-(6) from the final goods producers, equations (11)-(13) from the consumers, equations (16)-(19) from the intermediate goods producers, and equations (20)-(23) that must hold in equilibrium.

3. Simple Version with Convex Demand

In this section, we derive analytical results for the economy with $N = 2$, $F(K, L) = L$, and the more simple money demand equation:

$$(24) \quad \frac{M(s^t)}{\bar{P}(s^t)} = C(s^t)$$

in place of (12). This assumes an interest elasticity of 0. If we linearize (24), we get

$$(25) \quad m_t - \bar{p}_t = c_t.$$

Our derivations use the money demand equation, the pricing equation for the monopolists, the resource constraint, and the first order condition for consumers relating the wage rate to $-U_l/U_c$. The pricing equation for the group of monopolists (named i) that is currently adjusting prices is:

$$
\begin{aligned}
(26) \quad & \sum_{\tau} \sum_{s^\tau} Q(s^\tau | s^{t-1}) \Big\{ Y(i, s^\tau) + Y(s^\tau) \left[1 - \bar{P}(s^\tau) W(s^\tau) / P(i, s^{t-1}) \right] \\
& D_1 \left(\frac{P(i, s^{t-1}}{\bar{P}(s^\tau)} \int g_1 \left(\frac{Y(j, s^\tau)}{Y(s^\tau)}, \lambda(s^\tau) \right) \frac{Y(j, s^\tau)}{Y(s^\tau)}\, dj, \lambda(s^\tau) \right) g_1 \left(\frac{Y(i, s^\tau)}{Y(s^\tau)}, \lambda(s^\tau) \right) \Big\} = 0
\end{aligned}
$$

which follows from (17) with the unit cost equal to the wage rate, $V(i, s^t) = W(s^t)$.

Next, we assume that $\beta \approx 1$ and linearize (26). We'll do this in steps, starting with a rewriting of the equation so that variables are in logs:

$$
\begin{aligned}
0 \quad = \quad & \sum_\tau e^{q_\tau} \Big\{ e^{y_{i,\tau}} + e^{y_\tau} \left[1 - e^{\bar{p}_\tau + w_\tau - p_{i,t-1}}\right] \\
& \qquad D_1 \left(e^{p_{i,t-1} - \bar{p}_\tau} \int g_1(e^{y_{j,\tau} - y_\tau}, \lambda_\tau) e^{y_{j,\tau} - y_\tau} \, dj \right) g_1(e^{y_{i,\tau} - y_\tau}, \lambda_\tau) \Big\} \\
= \quad & \sum_\tau e^q e^y \Big\{ y_{i,\tau} + [(1 - e^w) y_\tau - e^w (\bar{p}_\tau + w_\tau - p_{i,t-1})] D_1(g_1(1,\lambda),\lambda) g_1(1,\lambda) \\
& \quad + (1 - e^w) \Big\{ D_{11}(g_1(1,\lambda),\lambda) g_1(1,\lambda) \\
& \qquad \cdot [g_1(1,\lambda)(p_{i,t-1} - \bar{p}_\tau) + (g_{11}(1,\lambda) + g_1(1,\lambda)) \left[\int y_{j,\tau} \, dj - y_\tau \right] + g_{12}(1,\lambda)\lambda_\tau] \\
& \qquad + D_{12}(g_1(1,\lambda),\lambda) g_1(1,\lambda) \lambda_\tau \\
& \qquad + D_1(g_1(1,\lambda),\lambda) [g_{11}(1,\lambda)(y_{i,\tau} - y_\tau) + g_{12}(1,\lambda)\lambda_\tau] \Big\} \Big\} \\
= \quad & \sum_\tau e^q e^y \Big\{ y_{i,\tau} + [(1 - e^w) y_\tau - e^w (\bar{p}_\tau + w_\tau - p_{i,t-1})](-\epsilon) \\
& \quad + (1 - e^w) \{ \chi\epsilon(p_{i,t-1} - \bar{p}_\tau) + y_{i,\tau} - y_\tau \} \\
& \quad + (1 - e^w) \{ -\chi(1 - \epsilon)(\int y_{j,\tau} \, dj - y_\tau) \} \\
& \quad + (1 - e^w) \{ \chi\epsilon g_{12}(1,\lambda)/g_1(1,\lambda) + D_{12}(g_1(1,\lambda),\lambda) g_1(1,\lambda) \\
& \qquad\qquad + D_1(g_1(1,\lambda),\lambda) g_{12}(1,\lambda) \} \lambda_\tau \Big\} \\
= \quad & \sum_\tau e^q e^y \Big\{ (1 + \epsilon)/\epsilon (y_{i,\tau} - y_\tau) + (1 - \epsilon + \chi)(p_{i,t-1} - \bar{p}_\tau) + (\epsilon - 1) w_\tau \\
& \qquad\qquad - \chi(1 - \epsilon)/\epsilon(\int y_{j,\tau} \, dj - y_\tau) + \psi \lambda_\tau \Big\}
\end{aligned}
\tag{27}
$$

where $\epsilon = -D_1(g_1(1), \lambda) g_1(1, \lambda)$, $\chi = -D_{11}(g_1(1,\lambda),\lambda) g_1(1,\lambda) / D_1(g_1(1,\lambda),\lambda)$, and

$$
\psi = (\chi - 1) \frac{g_{12}(1,\lambda)}{g_1(1,\lambda)} - \frac{D_{12}(g_1(1,\lambda),\lambda)}{D_1(g_1(1,\lambda),\lambda)}. \tag{28}
$$

Above, we have used the fact that $e^w = (\epsilon - 1)/\epsilon$ in the steady state. Below, we will also use the relation between D and g through $D(g_1(y,\lambda),\lambda) = y$ and hence

$$
\begin{aligned}
1 &= D_1(g_1(1,\lambda),\lambda)g_{11}(1,\lambda) \\
0 &= D_1(g_1(1,\lambda),\lambda)g_{12}(1,\lambda) + D_2(g_1(1,\lambda),\lambda) \\
0 &= D_{11}(g_1(1,\lambda),\lambda)g_{11}(1,\lambda)g_{12}(1,\lambda) + D_{12}(g_1(1,\lambda),\lambda)g_{11}(1,\lambda) + D_1(g_1(1,\lambda),\lambda)g_{112}(1,\lambda) \\
0 &= D_{11}(g_1(1,\lambda),\lambda)g_{11}(1,\lambda)g_{11}(1,\lambda) + D_1(g_1(1,\lambda),\lambda)g_{111}(1,\lambda) \qquad (29)
\end{aligned}
$$

at the steady state output ratio $y = 1$.

To further simplify the expression (27) derived from the pricing equation, we need to linearize (2) to get a relation between y and the y_j's as follows:

$$
\begin{aligned}
1 &= \int g(e^{y_{jt} - y_t}, \lambda_t)\, dj \\
&\approx \int g_1(1,\lambda)(y_{j,t} - y_t)\, dj + g_2(1,\lambda)\lambda_t + \text{constants}
\end{aligned}
$$

which, when rearranged, is

$$
y_t = \int y_{j,t}\, dj + \frac{g_2(1,\lambda)}{g_1(1,\lambda)}\lambda_t \equiv \int y_{j,t}\, dj + \kappa\lambda_t \qquad (30)
$$

where constants are ignored.

We also need to linearize the demand equation, which can be written as follows

$$
g_1(e^{y_{i,t} - y_t}, e^{\xi_t}) = e^{p_{i,t-1} - \bar{p}_t} \int g_1(e^{y_{jt} - y_t}, \lambda_t) e^{y_{jt} - y_t}\, dj \qquad (31)
$$

and implies

$$
\begin{aligned}
&g_{11}(1,\lambda)(y_{i,t} - y_t) + g_{12}(1,\lambda)\lambda_t \\
&\qquad = g_1(1,\lambda)(p_{i,t-1} - \bar{p}_t) + [g_{11}(1,\lambda) + g_1(1,\lambda)]\left[\int y_{j,t}\, dj - y_t\right] + g_{12}(1,\lambda)\lambda_t
\end{aligned}
$$

$$= g_1(1,\lambda)(p_{i,t-1} - \bar{p}_t) - [g_{11}(1,\lambda) + g_1(1,\lambda)]\kappa\lambda_t + g_{12}(1,\lambda)\lambda_t$$

when linearized. Rearranging this equation and using the definition of ϵ yields,

$$y_{i,t} - y_t \;=\; -\epsilon(p_{i,t-1} - \bar{p}_t) - (1-\epsilon)\kappa\lambda_t. \tag{32}$$

The last step in deriving the linearized pricing equation is to substitute (30) and (32) into (27) to get

$$\begin{aligned}
0 &= \sum_\tau e^q e^y \Big\{ (1+\epsilon)/\epsilon[-\epsilon(p_{i,t-1} - \bar{p}_\tau) - (1-\epsilon)\kappa\lambda_\tau] + (1-\epsilon+\chi)(p_{i,t-1} - \bar{p}_\tau) \\
&\qquad + (\epsilon - 1)w_\tau - \chi(1-\epsilon)/\epsilon[-\kappa\lambda_\tau] + \psi\lambda_\tau \Big\} \\
&= \sum_\tau e^q e^y \Big\{ (\chi - 2\epsilon)(p_{i,t-1} - \bar{p}_\tau) + (\epsilon - 1)w_\tau \\
&\qquad + [\chi\kappa(1-\epsilon)/\epsilon - (1-\epsilon^2)\kappa/\epsilon + \psi]\lambda_\tau \Big\} \\
&\propto \sum_\tau \Big\{ p_{i,t-1} - \bar{p}_\tau - \varphi w_\tau - \eta\lambda_\tau \Big\}
\end{aligned}$$

where

$$\eta = \frac{1}{2\epsilon - \chi}\left[\chi\kappa(1-\epsilon)/\epsilon - (1-\epsilon^2)\kappa/\epsilon + \psi\right] \tag{33}$$

If we used Calvo pricing instead, and allow for $\beta < 1$, then we linearize

$$\begin{aligned}
0 &= \sum_\tau (\beta\xi)^{\tau - t} e^{q_\tau} \Big\{ e^{y_{i,\tau}} + e^{y_\tau}\left[1 - e^{\bar{p}_\tau + w_\tau - p_{i,t-1}}\right] \\
&\qquad D_1\left(e^{p_{i,t-1} - \bar{p}_\tau} \int g_1(e^{y_{j,\tau} - y_\tau}, \lambda_\tau) e^{y_{j,\tau} - y_\tau}\, dj \right) g_1(e^{y_{i,\tau} - y_\tau}, \lambda_\tau) \Big\}
\end{aligned}$$

where ξ is the probability of being stuck. This implies,

$$0 \propto \sum_\tau (\beta\xi)^{\tau - t} \Big\{ p_{i,t-1} - \bar{p}_\tau - \varphi w_\tau - \eta\lambda_\tau \Big\} \tag{34}$$

and therefore

$$\text{(35)} \qquad \frac{1}{1-\beta\xi}p_{i,t-1} = E_{t-1}[x_t + \beta\xi x_{t+1} + \beta^2\xi^2 x_{t+2}\ldots]$$

where $x_t = \bar{p}_t + \varphi w_t + \eta\lambda_t$. We can rewrite this recursively as

$$\text{(36)} \qquad p_{i,t-1} = E_{t-1}[\beta\xi p_{i,t} + (1-\beta\xi)(\bar{p}_t + \varphi w_t + \eta\lambda_t)]$$

To simplify, we need to subtitutitue out $p_{i,t}$ from (36). We can do this by linearizing the zero-profit condition

$$\begin{aligned}
1 &= \int e^{p_{i,t-1}-\bar{p}_t} D\left(e^{p_{i,t-1}-\bar{p}_t}\int g_1(e^{y_{jt}-y_t},\lambda_t)e^{y_{jt}-y_t}\,dj,\lambda_t\right)\,di \\
&\approx \int\Big[p_{i,t-1}-\bar{p}_t + D_1(g_1(1,\lambda),\lambda)g_1(1,\lambda)(p_{i,t-1}-\bar{p}_t) \\
&\qquad +D_1(g_1(1,\lambda),\lambda)\{g_{11}(1,\lambda)+g_1(1,\lambda)\}\int(y_{jt}-y_t)\,dj \\
&\qquad +\{D_1(g_1(1,\lambda),\lambda)g_{12}(1,\lambda)+D_2(g_1(1,\lambda),\lambda)\}\lambda_t\Big]\,di \\
&= (1-\epsilon)(\int p_{i,t-1}\,di - \bar{p}_t) - \kappa(1-\epsilon)\lambda_t
\end{aligned}$$

Rearranging, we get

$$\text{(37)} \qquad \bar{p}_t = \int p_{i,t-1}\,di - \kappa\lambda_t.$$

Under Calvo pricing, (37) is written

$$\text{(38)} \qquad \bar{p}_t = (1-\xi)p_{i,t-1} + \xi\bar{p}_{t-1} - \kappa\lambda_t.$$

Using this to replace the p_{it}'s above, we get

$$\text{(39)} \qquad \bar{p}_t - \xi\bar{p}_{t-1} + \kappa\lambda_t = E_{t-1}\left[\beta\xi(\bar{p}_{t+1} - \xi\bar{p}_t + \kappa\lambda_{t+1}) + (1-\xi)(1-\beta\xi)(\bar{p}_t + \varphi w_t + \eta\lambda_t)\right]$$

Rearranging this equation and applying the law of iterated expectations yields:

$$\text{(40)} \qquad \bar{p}_t = \frac{1}{1+\beta}\bar{p}_{t-1} + E_t\left[\frac{\beta}{1+\beta}(\bar{p}_{t+1}+\kappa\lambda_{t+1})\right] + \frac{(1-\xi)(1-\beta\xi)}{\xi(1+\beta)}(\varphi w_t + \eta\lambda_t) - \kappa\lambda_t$$

Now consider the CES case for g:

(41) $g(y, \lambda) = \exp\{\theta(\lambda) \log y\}$

where $\theta(\lambda) = 1/(1 + \lambda)$. For this functional form,

$$
\begin{aligned}
g_1(y, \lambda) &= \theta/y g(y, \lambda) \\
g_2(y, \lambda) &= \theta' \log y g(y, \lambda) \\
g_{11}(y, \lambda) &= -\theta/y^2 g(y, \lambda) + \theta/y g_1(y, \lambda) \\
g_{12}(y, \lambda) &= \theta'/y g(y, \lambda) + \theta/y g_2(y, \lambda) \\
g_{111}(y, \lambda) &= 2\theta/y^3 g(y, \lambda) - 2\theta/y^2 g_1(y, \lambda) + \theta/y g_{11}(y, \lambda) \\
g_{112}(y, \lambda) &= -\theta'/y^2 g(y, \lambda) + \theta' g_1(y, \lambda) - \theta/y^2 g_2(y, \lambda) + \theta/y g_{12}(y, \lambda)
\end{aligned}
$$

Evaluated at a steady state, these expressions are

$$
\begin{aligned}
g(1, \lambda) &= 1 \\
g_1(1, \lambda) &= \theta \\
g_2(1, \lambda) &= 0 \\
g_{11}(1, \lambda) &= \theta(\theta - 1) \\
g_{12}(1, \lambda) &= \theta' \\
g_{111}(1, \lambda) &= \theta(\theta - 1)(\theta - 2) \\
g_{112}(1, \lambda) &= \theta'(2\theta - 1)
\end{aligned}
$$

where we have suppressed the arguments of $\theta(\lambda)$ and $\theta'(\lambda)$. Therefore

$$
D_{11} = -\frac{(\theta - 2)}{\theta^2(\theta - 1)^2}
$$

$$
\begin{aligned}
D_{12} &= -\frac{\theta'(1+\theta)}{\theta^2(\theta-1)^2} \\
\kappa &= 0 \\
\epsilon &= \frac{1}{\theta-1} \\
\chi &= \frac{\theta-2}{\theta-1} \\
\eta &= \frac{1}{2\epsilon-\chi}\left[(\chi-1)\frac{\theta'}{\theta}-\frac{D_{12}}{D_1}\right] = \frac{-\theta'}{2-\chi/\epsilon} = \frac{1}{1+\lambda}
\end{aligned}
$$

where we have suppressed the arguments of the derivatives of D.