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NOTES: Openness, Technology Capital, and Development†

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† The views expressed herein are those of the author and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1. Introduction

In these notes, we provide additional details for the results reported in our paper, 'Openness, Technology Capital, and Development.'

2. Aggregate Production with Foreign Technology Capital

We start by motivating the aggregate production technology that we use in the paper. Consider maximizing output in a country that has a composite input Z (e.g., a composite of capital and labor), technology capital of domestic multinationals M_d, technology capital of foreign multinationals M_f, and N locations. Let Y be the maximal output,

$$\begin{gathered} Y = \max_{z_d, z_f} ANM_d \left(z_d\right)^{1-\phi} + A\sigma NM_f \left(z_f\right)^{1-\phi} \\ \text{subject to } \; NM_d z_d + NM_f z_f \leq Z \end{gathered}$$

where NM_d are the number of location-technologies where z_d can be used and NM_f are the number where z_f can be used. The parameter A being the level of technology in the country and $\sigma \leq 1$ implies there can be a cost on foreign producers. Writing out the Lagrangian and differentiating with respect to z_d and z_f implies

$$z_d = \frac{Z}{NM_d + \sigma^{\frac{1}{\phi}} NM_f}$$

$$z_f = \frac{\sigma^{\frac{1}{\phi}} Z}{NM_d + \sigma^{\frac{1}{\phi}} NM_f}$$

and therefore,

$$Y = A \left(NM_d + \sigma^{\frac{1}{\phi}} NM_f\right)^{\phi} Z^{1-\phi}.$$

There is another way to write this problem. Assume that fraction ω of the foreign technology capital can be used domestically. Then the maximal output is

$$\begin{gathered} Y = \max_{z_d, z_f} ANM_d \left(z_d\right)^{1-\phi} + AN\omega M_f \left(z_f\right)^{1-\phi} \\ \text{subject to } \; NM_d z_d + N\omega M_f z_f \leq Z. \end{gathered}$$

Writing out the Lagrangian and differentiating with respect to z_d and z_f implies $z_d = z_f = Z/(NM_d + N\omega M_f)$ and

$$Y = A \left(NM_d + N\omega M_f\right)^{\phi} Z^{1-\phi} = A \left(N\hat{M}\right)^{\phi} Z^{1-\phi}$$

where $\hat{M}$ is the effective technology capital used in domestic production. Notice that if $\omega = 0$, then no foreign technology capital can be used and $\hat{M} = M_d$. If all foreign technology capital can be used ($\omega = 1$), then $\hat{M} = M_d + M_f$.

Next, consider adding capital and labor. Let $Z = K^\alpha L^{1-\alpha}$, where K is capital and L is labor. In this case, aggregate output in country i is

$$Y_i = A_i \left(N_i \hat{M}_i\right)^\phi \left(K_i^\alpha L_i^{1-\alpha}\right)^{1-\phi}$$

where $\hat{M}_i = M_i + \omega_i \sum_{j \neq i} M_j$.

3. Our Model Economy

There are I countries. Households in country i own technology capital of companies incorporated in country i, M_i, and nontechnology capital used in i, K_i. They choose investments in these stocks, X_{im} and X_{ik}, consumption C_i and labor L_i to maximize

$$\max \sum_t \beta^t U\left(C_{it}/N_{it}, L_{it}/N_{it}\right) N_{it} \tag{3.1}$$

subject to

$$\begin{aligned}
&C_{it} + X_{ikt} + X_{imt} + NX_{it} = Y_{it} \\
&Y_{it} = A_{it} \left(N_{it} \hat{M}_{it}\right)^\phi \left(K_{it}^\alpha L_{it}^{1-\alpha}\right)^{1-\phi} \\
&\hat{M}_{it} = M_{it} + \omega_{it} \sum_{j \neq i} M_{jt} \\
&K_{i,t+1} = \left(1 - \delta_k\right) K_{it} + X_{ikt} \\
&M_{i,t+1} = \left(1 - \delta_m\right) M_{it} + X_{imt} \\
&NX_{it} + \sum_{j \neq i} r_{it}^j M_{it} - \sum_{j \neq i} r_{jt}^i M_{jt} = 0 \\
&M_{it} \geq 0 \;\; \forall i \\
&N_{it} = \left(1 + \gamma_N\right)^t N_{i0} \\
&K_{i0}; M_{i0}; \{A_{it}, \omega_{it}\} \; \forall i, t; \{M_{jt}\} \; \forall j \neq i, t; \{r_{it}^j\} \; \forall i, j, t \text{ given.}
\end{aligned}$$

The price r_{it}^j is the rental rate paid in t by multinationals in j for technology capital from country i. So, the rents on M_i from country j are equal to $r_{it}^j M_{it}$. In equilibrium, these prices are related to the marginal products of the M's,

$$r_{it}^j = \phi \omega_{jt} Y_{jt} / \hat{M}_{jt}.$$

We derive two types of results for this model economy. First, we analytically derive steady state results for the case where labor is inelastically supplied, there is no growth, and ω is independent of i. Second, we compute transition paths relaxing these assumptions. On the transition path, we allow for changes in $\sigma_{it} = \omega_{it}^{\phi}$ and A_{it}.

4. Steady State Results

This section derives formulas for the following cases of our model economy with labor inelastically supplied, no growth, and common ω. We then extend the results to allow for elastically supplied labor and growth.

- Proposition 1 concerns output per capita (Y_i/N_i) in a two-country world when only populations differ and $N_1 > N_2$;
- Proposition 2 concerns output per capita in a I-country world when populations differ and $N_1 > \cdots > N_I$;
- Proposition 3 concerns output per effective person in a I-country world when populations and TFPs differ and $\mathcal{A}_1 N_1 > \cdots > \mathcal{A}_I N_I$.

At the end of the section, we specify asset ownership in order to derive predictions for consumption and then apply our formulas as we vary σ.

4.1. Proposition 1.

There are two countries, that both have the same degree of openness. We assume that $\omega < 1$ (and hence $\sigma < 1$) and consider limiting economies as ω approaches one. At $\omega = 1$, there is an indeterminacy.

There are two categories of steady state: (i) both countries have positive stocks of technology capital; (ii) only the country with the largest population has a positive stock of technology capital. The category depends on the size of the parameter ω, which is the amount of foreign technology capital allowed into any country. Let $\omega^* = (N_2/N_1)^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}$. If $\omega < \omega^*$, both countries have positive stocks of technology capital and

$$\frac{Y_i}{N_i} \propto \left[N_i\left(1+\omega\right)\right]^{\frac{\phi}{(1-\alpha)(1-\phi)}}, \quad i = 1, 2. \tag{4.1}$$

In (4.1) and throughout these notes, we assume that the constants of proportionality do not depend on ω or the vector of populations. If $\omega > \omega^*$, then only the country with the largest population has a stock of technology capital and

$$\frac{Y_1}{N_1} \propto \left[N_1 + \omega^{\frac{\phi}{1-\alpha(1-\phi)}} N_2\right]^{\frac{\phi}{(1-\alpha)(1-\phi)}}$$

$$\frac{Y_2}{N_2} \propto \left[N_1 + \omega^{\frac{\phi}{1-\alpha(1-\phi)}} N_2\right]^{\frac{\phi}{(1-\alpha)(1-\phi)}} \omega^{\frac{\phi}{1-\alpha(1-\phi)}}. \tag{4.2}$$

Proof. The steady state of our economy is the solution of the following equations:

$$\rho + \delta_k = \alpha(1-\phi)\frac{Y_i}{K_i}, \quad i = 1, 2 \tag{4.3}$$

$$\rho + \delta_m \geq \phi\left(\frac{Y_1}{M_1 + \omega M_2} + \omega\frac{Y_2}{M_2 + \omega M_1}\right), \quad \text{with } = \text{if } M_1 > 0 \tag{4.4}$$

$$\rho + \delta_m \geq \phi\left(\frac{Y_2}{M_2 + \omega M_1} + \omega\frac{Y_1}{M_1 + \omega M_2}\right), \quad \text{with } = \text{if } M_2 > 0 \tag{4.5}$$

$$L_i = N_i, \quad i = 1, 2 \tag{4.6}$$

$$Y_i = AN_i^{\phi}(M_i + \omega M_{-i})^{\phi}\left(K_i^{\alpha}L_i^{1-\alpha}\right)^{1-\phi}, \quad i = 1, 2 \tag{4.7}$$

where M_{-i} is the foreign capital used in country i.

Equations (4.3) equate the rental price to the return on K. Equations (4.4) and (4.5) are not standard. Here, the return is equal to the sum of domestic and foreign returns because the same technology capital can be used simultaneously in domestic and foreign production. Equation (4.6) is the assumption that labor is inelastic, and (4.7) is the production technology.

From (4.3) we have

$$K_i \propto Y_i, \quad i = 1, 2. \tag{4.8}$$

Using (4.8) and (4.6), equation (4.7) implies

$$Y_i \propto N_i(M_i + \omega M_j)^{\frac{\phi}{1-\alpha(1-\phi)}}. \tag{4.9}$$

Next, we need to derive expressions for the M's. We know from (4.4) and (4.5) that if the inequality constraints on the M's are not binding, then the ratios of output relative to the total technology capital used are equal, that is

$$\frac{Y_1}{M_1 + \omega M_2} = \frac{Y_2}{M_2 + \omega M_1} = \frac{1}{1+\omega}\left(\frac{\rho + \delta_m}{\phi}\right). \tag{4.10}$$

Using (4.10) along with the production function (4.9),

$$\begin{aligned}
\frac{Y_i}{N_i} &= \left(A\kappa^{\alpha(1-\phi)}\right)^{\frac{1}{(1-\alpha)(1-\phi)}}\left((1+\omega)\,N_i\frac{Y_i}{N_i}\right)^{\frac{\phi}{1-\alpha(1-\phi)}} \\
&\propto \left((1+\omega)\,N_i\frac{Y_i}{N_i}\right)^{\frac{\phi}{1-\alpha(1-\phi)}} \qquad (4.11)
\end{aligned}$$

where κ is the capital output ratio. Equation (4.11) is the first result of the proposition. (See (4.1).)

Another finding is that the constraint $M_2 \geq 0$ will be binding for sufficiently large ω. To see this, solve for M_1 and M_2 using the relations in (4.10):

$$\begin{aligned}
M_1 &= \frac{\phi}{\rho+\delta_m}\frac{Y_1-\omega Y_2}{1-\omega} \\
M_2 &= \frac{\phi}{\rho+\delta_m}\frac{Y_2-\omega Y_1}{1-\omega}. \qquad (4.12)
\end{aligned}$$

For $\omega = 0$, M_i is proportional to Y_i as in the case with tangible capital.

Lemma. There is a unique ω such that $M_2 = 0$.

Proof. Substitute the equilibrium output in the unconstrained case from (4.1) into (4.12) to get

$$M_2 \propto \frac{(1+\omega)^{\frac{\phi}{(1-\alpha)(1-\phi)}}}{1-\omega}\left(N_2^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} - \omega N_1^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}\right) \qquad (4.13)$$

which can be written $M_2 = a(\omega)(1-\omega x)$ where x is greater than 1 and $a(\omega)$ is strictly positive for all ω. Thus, M_2 has a unique zero at $\omega = \omega^*$:

$$\omega^* = \left(\frac{N_2}{N_1}\right)^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}. \qquad (4.14)$$

Now consider the case $\omega > \omega^*$ where the nonnegativity constraint on M_2 binds. In this case, (4.4) can be rewritten,

$$\rho+\delta_m = \phi\,(Y_1+Y_2)\,/M_1 \qquad (4.15)$$

so that technology capital in country 1 (which is the total capital stock) is proportional to world output. Using (4.15) in (4.9), we get

$$\begin{aligned}
Y_1 &\propto\ N_1\,(Y_1+Y_2)^{\frac{\phi}{1-\alpha(1-\phi)}} \\
Y_2 &\propto\ N_2\omega^{\frac{\phi}{1-\alpha(1-\phi)}}\,(Y_1+Y_2)^{\frac{\phi}{1-\alpha(1-\phi)}}.
\end{aligned}$$

We sum these expressions for outputs and solve for world output $Y_1 + Y_2$. The result can be substituted back in to yield (4.2). ∎

If we did not constrain the openness parameters to be the same, the result is complicated by the fact that both size and openness affect the relative productivities. To see this, assume that $\omega_1 \neq \omega_2$. If the parameters are sufficiently small, then the nonnegativity constraints on M_i don't bind and the solution has

$$\frac{Y_i}{N_i} = a\left(N_i \frac{1-\omega_i\omega_{-i}}{1-\omega_{-i}}\right)^{\frac{\phi}{1-\alpha(1-\phi)}}$$

$$M_i = (1-\omega_i\omega_{-i})^{\frac{\phi}{(1-\alpha)(1-\phi)}}\left(\frac{\phi a}{\rho+\delta_m}\right)\left[\left(\frac{N_i}{1-\omega_{-i}}\right)^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} - \omega_i\left(\frac{N_{-i}}{1-\omega_i}\right)^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}\right]$$

where $a = (A\kappa^{\alpha(1-\phi)})^{1/[(1-\alpha)(1-\phi)]}$ If $\omega_1 < \omega_2$, then the solution involves a cutoff for ω_2: if ω_2 exceeds this cutoff, then $M_2 = 0$. If $\omega_1 > \omega_2$, then the relative population sizes and relative openness both play a role in determining whether country 1 or country 2 has their technology capital stock go to zero.

4.2. Proposition 2.

Next we generalize the result of proposition 1 to I countries. In our proof, we assume that (1) technology capital investment increases with population so that we can order countries by size, and (2) the cut-off is country n with $M_i > 0$ for $i = 1, \ldots, n$ and $M_i = 0$ for $i = n+1, \ldots, I$. Assumption (1) is actually a result, but we don't prove it here. We simply characterize the steady state given (1) and (2) are true and then show that it is consistent with (1) and (2).

We sort the populations of the I countries as follows: $N_1 > N_2 > \ldots > N_I$. We can characterize the steady state per capita output, given ω, for countries 1 through n with positive stocks of technology capital and countries $n+1$ through I with zero. In this case, the steady state per capita outputs and world stock of technology capital are given by:

$$\frac{Y_i}{N_i} \propto \left(\frac{N_i^{\mu}}{\sum_{j=1}^{n} N_j^{\mu}}\left(1+\omega\left(n-1\right)\right)M_w\right)^{\theta}, \quad i = 1, \ldots, n \tag{4.16}$$

$$\frac{Y_i}{N_i} \propto \left(\omega M_w\right)^{\theta}, \quad i = n+1, \ldots, I \tag{4.17}$$

$$M_w \propto \left((1+\omega(n-1))^{\theta} \left[\sum_{i=1}^{n} N_i^{\mu} \right]^{\frac{1}{\mu}} + \omega^{\theta} \sum_{i=n+1}^{I} N_i \right)^{\mu} \tag{4.18}$$

where $M_w = \sum_i M_i$ is the world stock of technology capital and

$$\theta = \frac{\phi}{1-\alpha(1-\phi)}, \qquad \mu = \frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}. \tag{4.19}$$

Proof. For convenience of notation, let $\hat{M}_i = M_i + \omega \sum_{j \neq i} M_j$ be the effective technology capital for country i. In the system of steady state equations, replace (4.4)–(4.5) for the two-country case with the following I equations

$$\begin{aligned}
\rho + \delta_m &= \phi \left(\frac{Y_1}{\hat{M}_1} + \omega \frac{Y_2}{\hat{M}_2} + \ldots + \omega \frac{Y_n}{\hat{M}_n} + \frac{Y_{n+1} + \ldots + Y_I}{M_w} \right) \\
\rho + \delta_m &= \phi \left(\omega \frac{Y_1}{\hat{M}_1} + \frac{Y_2}{\hat{M}_2} + \ldots + \omega \frac{Y_n}{\hat{M}_n} + \frac{Y_{n+1} + \ldots + Y_I}{M_w} \right) \\
&\vdots \\
\rho + \delta_m &= \phi \left(\omega \frac{Y_1}{\hat{M}_1} + \omega \frac{Y_2}{\hat{M}_2} + \ldots + \frac{Y_n}{\hat{M}_n} + \frac{Y_{n+1} + \ldots + Y_I}{M_w} \right)
\end{aligned} \tag{4.20}$$

where the remaining equations are: $M_{n+1} = M_{n+2} = \ldots = M_I = 0$.

With $\omega < 1$ the same for all countries, it is easy to see from the equations in (4.20) that

$$\frac{Y_1}{\hat{M}_1} = \frac{Y_2}{\hat{M}_2} = \ldots = \frac{Y_n}{\hat{M}_n} = \frac{Y_1 + \ldots + Y_n}{M_w(1+\omega(n-1))}. \tag{4.21}$$

Since only the first n technology capital stocks are positive, we have $M_w = M_1 + \ldots M_n$.

As in the two-country case, output in each country can be written in terms of population and technology capital used in the country,

$$Y_i \propto N_i \hat{M}_i^{\phi/[1-\alpha(1-\phi)]}. \tag{4.22}$$

To derive an expression for per-capita output, we need $\hat{M}_i$ for all countries. This is done in several steps.

The first step involves computing the world stock M_w. Using the results in (4.21) we can rewrite the first equation in (4.20) as

$$\begin{aligned}
(\rho + \delta_m)/\phi &= (1+\omega(n-1)) \frac{Y_1 + \ldots Y_n}{M_w(1+\omega(n-1))} + \frac{Y_{n+1} + \ldots Y_I}{M_w} \\
&= \frac{Y_1 + \ldots Y_n}{M_w} + \frac{Y_{n+1} + \ldots Y_I}{M_w} \\
&\equiv \frac{Y_u + Y_c}{M_w}
\end{aligned} \tag{4.23}$$

where Y_u is the sum of the output of the countries with no binding constraint ($M_i > 0$) and Y_c is the sum of the output of the countries with a binding constraint ($M_i = 0$).

Next, we write the sum of the outputs as functions of known populations and M_w, which we use to boil the problem down to one equation in one unknown: M_w. Equation (4.21) implies

$$\hat{M}_i = (1 + \omega (n-1)) \frac{M_w Y_i}{\sum_{j=1}^{n} Y_j}, \quad i = 1, \ldots n \tag{4.24}$$

which can be substituted into (4.22):

$$\begin{aligned} Y_i &\propto \; N_i \left((1 + \omega (n-1)) \frac{M_w Y_i}{\sum_{j=1}^{n} Y_j} \right)^{\frac{\phi}{1-\alpha(1-\phi)}} \\ &\propto \; N_i^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} \left((1 + \omega (n-1)) \frac{M_w}{\sum_{j=1}^{n} Y_j} \right)^{\frac{\phi}{(1-\alpha)(1-\phi)}}, \quad i = 1, \ldots n. \end{aligned} \tag{4.25}$$

Summing the Y_i's for $i = 1, \ldots n$ and solving for Y_u yields

$$Y_u \propto \; \left((1 + \omega (n-1)) M_w \right)^{\frac{\phi}{1-\alpha(1-\phi)}} \left[\sum_{i=1}^{n} N_i^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} \right]^{\frac{(1-\alpha)(1-\phi)}{1-\alpha(1-\phi)}}. \tag{4.26}$$

Substituting Y_u (4.26) in for $\sum_j Y_j$ in (4.25) and dividing by N_i yields

$$\frac{Y_i}{N_i} \propto \left(\frac{N_i^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}}{\sum_{j=1}^{n} N_j^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}}} (1 + \omega (n-1)) M_w \right)^{\frac{\phi}{1-\alpha(1-\phi)}} \tag{4.27}$$

which is the same as (4.16) above. Notice that the output is increasing in N_i. Since $\hat{M}_i$ is proportional to Y_i, then the result is consistent with our original assumption that technology capital is also increasing in N_i.

Next, consider the constrained countries indexed by $n+1, \ldots, I$. As in the case of the unconstrained, we can substitute the $\hat{M}_i$'s into (4.22) to get output. Since $\hat{M}_i = \omega M_w$, the per capita output of the constrained countries is

$$\frac{Y_i}{N_i} \propto \; (\omega M_w)^{\frac{\phi}{1-\alpha(1-\phi)}}, \quad i = n{+}1, \ldots I \tag{4.28}$$

which is the result in (4.17).

Expressions for Y_u and Y_c can be used in (4.23) to derive an expression for M_w in terms of country populations. The expression for Y_u is given in (4.26). The expression for Y_c is:

$$Y_c \propto \; (\omega M_w)^{\frac{\phi}{1-\alpha(1-\phi)}} \sum_{i=n+1}^{I} N_i. \tag{4.29}$$

Substituting these into (4.23) yields,

$$M_w \propto \; \Bigg((1+\omega\,(n-1))^{\frac{\phi}{1-\alpha(1-\phi)}} \left[\sum_{i=1}^{n} N_i^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} \right]^{\frac{(1-\alpha)(1-\phi)}{1-\alpha(1-\phi)}} + \omega^{\frac{\phi}{1-\alpha(1-\phi)}} \sum_{i=n+1}^{I} N_i \Bigg)^{\frac{1-\alpha(1-\phi)}{(1-\alpha)(1-\phi)}} \tag{4.30}$$

which is the same as (4.30). ∎

To check that we made the right assumption about which countries were constrained $(i > n)$, we can evaluate M_{n+1} using the formulas above for the unconstrained technology capital stocks to see if it is less than or equal to zero.

To derive the cut-off ω that implies $M_n > 0$ while $M_{n+1} = 0$, we can equate the expression for the unconstrained output and the constrained output for $i = n+1$, because this is the point where M_{n+1} is just constrained. This implies the following cut-off for ω:

$$\omega^* = \frac{N_{n+1}^{\mu}}{\sum_{j=1}^{n+1} N_j^{\mu} - nN_{n+1}^{\mu}}, \qquad \mu = \frac{1-\alpha\,(1-\phi)}{(1-\alpha)\,(1-\phi)}. \tag{4.31}$$

4.3. Proposition 3

We now generalize this result by allowing for differences in TFP. To make the math simpler, we replace (4.7) with

$$Y_i = (\mathcal{A}_i N_i)^{\phi}\,(M_i + \omega M_{-i})^{\phi}\, K_i^{\alpha(1-\phi)}\,(\mathcal{A}_i L_i)^{(1-\alpha)(1-\phi)}$$

and let TFP be $A_i = \mathcal{A}_i^{1-\alpha(1-\phi)}$. Our notion of country size is now $\mathcal{A}N$.

Assume there are I countries with $\mathcal{A}_1 N_1 > \mathcal{A}_2 N_2 > ... > \mathcal{A}_I N_I$. As before, we assume that countries 1 through n have positive stocks of technology capital while countries $n+1$

through I are constrained with stocks equal to zero. In this case, the steady state per capita outputs and world stock of technology capital are given by:

$$\frac{Y_i}{\mathcal{A}_i N_i} \propto \left(\frac{(\mathcal{A}_i N_i)^{\mu}}{\sum_{j=1}^{n} (\mathcal{A}_j N_j)^{\mu}} \left(1 + \omega\left(n-1\right)\right) M_w \right)^{\theta}, \quad i = 1, \ldots, n \tag{4.32}$$

$$\frac{Y_i}{\mathcal{A}_i N_i} \propto \left(\omega M_w\right)^{\theta}, \quad i = n+1, \ldots, I \tag{4.33}$$

$$M_w \propto \left(\left(1 + \omega\left(n-1\right)\right)^{\theta} \left[\sum_{i=1}^{n} (\mathcal{A}_i N_i)^{\mu} \right]^{\frac{1}{\mu}} + \omega^{\theta} \sum_{i=n+1}^{I} \mathcal{A}_i N_i \right)^{\mu} \tag{4.34}$$

where $M_w = \sum_i M_i$ is the world stock of technology capital as before and θ, μ are defined in (4.19).

Proof. After substituting out the tangible capital-output ratios and labor (using (4.3) and (4.6)), the set of equations have $\mathcal{A}$ and N appearing everywhere as a product. Therefore, in the proof of Proposition 2, we replace populations N_i for each i with size $\mathcal{A}_i N_i$. Everything goes through as before. ∎

4.4. Determining Consumption

To derive expressions for consumption, we assume that (*i*) households in i own K_i and M_i and (*ii*) the current account balance is zero (so net exports plus net factor income is zero). In this case, we append to our earlier system the following equations:

$$C_i + \delta_k K_i + \delta_m M_i + NX_i = Y_i$$

$$NX_i + \sum_{j \neq i} R_i^j - \sum_{j \neq i} R_j^i = 0$$

$$R_i^j \equiv \phi \omega Y_j M_i / \left(M_j + \omega M_{-j}\right)$$

$$R_j^i \equiv \phi \omega Y_i M_j / \left(M_i + \omega M_{-i}\right)$$

where $M_{-i} = \sum_{j \neq i} M_j$. The term R_i^j is the rent on M_i from country j.

We start with the two-country case with $A = \mathcal{A}^{1-\alpha(1-\phi)}$ constant. First, assume that ω is such that $M_2 > 0$. Then,

$$C_1 + \delta_k a_k Y_1 + \delta_m a_m \frac{Y_1 - \omega Y_2}{1 - \omega} + \phi \omega \left(\frac{Y_1 M_2}{M_1 + \omega M_2} - \frac{Y_2 M_1}{M_2 + \omega M_1} \right) = Y_1$$

$$C_2 + \delta_k a_k Y_2 + \delta_m a_m \frac{Y_2 - \omega Y_1}{1-\omega} + \phi\omega\left(\frac{Y_2 M_1}{M_2 + \omega M_1} - \frac{Y_1 M_2}{M_1 + \omega M_2}\right) = Y_2$$

where $a_k = \alpha(1-\phi)/(\rho+\delta_k)$ and $a_m = \phi/(\rho+\delta_m)$. Note that summing the consumptions yields implies that the world-wide resource constraint holds,

$$C_1 + C_2 = (1 - \delta_k a_k - \delta_m a_m)(Y_1 + Y_2).$$

We rewrite per capita consumption in terms of the world populations (or size if we allow TFPs to vary) as follows:

$$\frac{C_i}{N_i} = \left[1 - \delta_k a_k - (\delta_m a_m - \omega\phi)\frac{1}{1-\omega}\right]\frac{Y_i}{N_i} + (\delta_m a_m - \phi)\frac{\omega}{1-\omega}\frac{Y_{-i}}{N_i}$$

$$= a_y\Bigg\{\left[1 - \delta_k a_k - (\delta_m a_m - \omega\phi)\frac{1}{1-\omega}\right] N_i^{\frac{\phi}{(1-\alpha)(1-\phi)}}$$

$$+ (\delta_m a_m - \phi)\frac{\omega}{1-\omega}\frac{N_{-i}}{N_i} N_{-i}^{\frac{\phi}{(1-\alpha)(1-\phi)}}\Bigg\}(1+\omega)^{\frac{\phi}{(1-\alpha)(1-\phi)}}$$

where a_y is the constant in front of equilibrium output.

Next consider the region of ω such that $M_2 = 0$. In this case,

$$M_1 = \frac{\phi}{\rho + \delta_m}(Y_1 + Y_2)$$

and

$$C_1 = (1 - \delta_k a_k \delta_m a_m) Y_1 + (\phi - \delta_m a_m) Y_2$$

$$C_2 = (1 - \delta_k a_k - \phi) Y_2.$$

The ratio of per-capita consumptions in this constrained region (with $M_2 = 0$) is equal to

$$\frac{c_1}{c_2} = \frac{(1 - \delta_k a_k - \delta_m a_m)\,\omega^{-\frac{\phi}{1-\alpha(1-\phi)}} + \phi - \delta_k a_k}{1 - \delta_k a_k - \phi}$$

where $c_i = C_i/N_i$. This is true because the ratio of per capita outputs y_1/y_2 is equal to $\omega^{-\phi/(1-\alpha(1-\phi))}$.

What happens to country 2 when they integrate with country 1? When closed, country 2 has per capita equal to

$$c_2^c = a_y (1 - \delta_k a_k - \delta_m a_m) N_2^{\frac{\phi}{(1-\alpha)(1-\phi)}}.$$

If country 2 joins country 1 in an economic union but finds it optimal to set $M_2 = 0$, then

$$c_2^o = a_y \left(1 - \phi - \delta_k a_k\right) \left(N_1 + \omega^{\frac{\phi}{(1-\alpha)(1-\phi)}} N_2\right)^{\frac{\phi}{(1-\alpha)(1-\phi)}} \omega^{\frac{\phi}{1-\alpha(1-\phi)}}.$$

This is the case if ω is large enough to make it worthwhile to use country 1's technology capital. As ω approaches 1, country 2's per capita consumption approaches:

$$c_2^o \left(\omega \to 1\right) = a_y \left(1 - \delta_k a_k - \phi\right) \left(N_1 + N_2\right)^{\frac{\phi}{(1-\alpha)(1-\phi)}}.$$

Next, we parameterize the model to see how large the gains to opening are.

4.5. Applications

We now apply the formulas above. We choose parameters to get a tangible capital output ratio of 3, a technology capital output ratio of 1/2, a labor share of 66 percent and a real interest rate of 4 percent. Specifically we use: $\alpha = .3$, $\delta_k = .053$, $\delta_m = .1$, $\rho = .04$, and $\phi = .07$.

The United States and Canada. Suppose that country 1 (the United States) is 10 times bigger than country 2 (Canada). The critical degree of openness $\sigma^* = (\omega^*)^\phi$ in this case is 0.84 $(=.078^{.07})$. If $\sigma < \sigma^*$, then $y_1/y_2 = 10^{.108} = 1.28$ at all points $\sigma \in [0, \sigma^*]$ where y_i is per capital output. If $\sigma > \sigma^*$, then $y_1/y_2 = \sigma^{-1.39}$, which monotonically decreases to 1 as σ approaches 1. When completely integrated, the gain in per capita output for Canada is $11^{.108}$=1.294.

What about consumption? In this case, the ratio of the small country's per capita consumption when completely open to the per capita consumption when completely closed is $c_2^o(\omega \to 1)/c_2^c = 1.26$, which is close to the relative productivities.

The European Union. Suppose we model Europe as I similar sized countries with the same levels of TFP and N_i normalized to 1. The per capita outputs of economically integrated European countries is $I^{\cdot}108$ times their per capita output when closed. For example, if $I = 10$, the per-capita gain is 28 percent. Thus, being a large union of countries is like being a large country.

Europe and the United States. Now suppose that 10 European countries with $N_i = 1$ and the United States with $N_i = 10$ integrate. In this case, only U.S. technology capital is nonzero if σ is sufficiently large. The critical point here is again $\sigma^* = 0.84$. (as in the example of Canada versus the United States). The relative per capita output of the United States and any of the Europeans is 1.28 for all values $\sigma \in [0, \sigma^*]$. For $\sigma > \sigma^*$, the degree of openness does affect the result. As σ approaches 1, the per capita output of the United States relative to any of the European countries approaches 1. When the union is formed, it has a population of size 20 and per capita output equal to $20^{\cdot 108}$ or 1.38 times that of the closed European country prior to joining.

5. Transition Results

We turn now to computing balanced growth results and transitional paths. We allow for growth, elastically supplied labor, and country-specific measures of openness. We also impose nonnegativity restrictions on investments in both technology capital and tangible capital.

We start by deriving the growth rate of output on the balanced growth path. Using the production relation in the problem (3.1), we can relate the growth rate of total output to the growth rates of TFP (γ_A) and population (γ_N):

$$
\begin{aligned}
1+\gamma_Y &= (1+\gamma_A)(1+\gamma_N)^{\phi}(1+\gamma_Y)^{\phi}(1+\gamma_Y)^{\alpha(1-\phi)}(1+\gamma_N)^{(1-\alpha)(1-\phi)} \\
&= (1+\gamma_A)^{\frac{1}{(1-\alpha)(1-\phi)}}(1+\gamma_N)^{\frac{\phi\alpha}{(1-\alpha)(1-\phi)}}
\end{aligned}
$$

using the fact that K and M grow at the same rate as Y and L grows at the same rate as N. Later, we will use these growth rates to transform all variables and make them stationary.

Next, we need the dynamic first-order conditions of our model which are given by

$$
\begin{aligned}
&\beta^t N_{it}/C_{it} = p_{it} \\
&\beta^t \psi L_{it} = p_{it}(1-\alpha)(1-\phi)(1-L_{it}/N_{it})Y_{it} \\
&p_{it} = p_{i,t+1}\left[1-\delta_k+\alpha(1-\phi)\frac{Y_{i,t+1}}{K_{i,t+1}}\right] \\
&p_{it} = p_{i,t+1}\left[1-\delta_m+\phi\frac{Y_{i,t+1}}{\hat{M}_{i,t+1}}+\sum_{j\neq i}\omega_{j,t+1}\frac{Y_{j,t+1}}{\hat{M}_{j,t+1}}\right] \\
&Y_{it} = A_{it}N_{it}^{\phi}(M_{it}+\omega_{it}M_{-it})^{\phi}(K_{it})^{\alpha(1-\phi)}L_{it}^{(1-\alpha)(1-\phi)}
\end{aligned}
$$

$$X_{ikt} = K_{i,t+1} - (1-\delta_k) K_{i,t}, \quad X_{ikt} \geq 0$$

$$X_{imt} = M_{i,t+1} - (1-\delta_m) M_{i,t}, \quad X_{imt} \geq 0.$$

The notation M_{-it} means the total technology capital outside of country i.

We want to rewrite the first-order equations so that all variables are stationary. Define the following intermediate variables,

$$c_{it} = \frac{C_{it}}{N_{it}(1+\gamma_y)^t} = \frac{C_{it}}{N_{io}(1+\gamma_Y)^t}$$

$$x_{ikt} = \frac{X_{ikt}}{N_{it}(1+\gamma_y)^t} = \frac{X_{ikt}}{N_{io}(1+\gamma_Y)^t}$$

$$k_{it} = \frac{K_{it}}{N_{it}(1+\gamma_y)^t} = \frac{K_{it}}{N_{io}(1+\gamma_Y)^t}$$

$$y_{it} = \frac{Y_{it}}{N_{it}(1+\gamma_y)^t} = \frac{Y_{it}}{N_{io}(1+\gamma_Y)^t}$$

$$l_{it} = \frac{L_{it}}{N_{it}}$$

$$x_{imt} = \frac{X_{imt}}{(1+\gamma_Y)^t}$$

$$m_{it} = \frac{M_{it}}{(1+\gamma_Y)^t}$$

$$a_{it} = \frac{A_{it}}{(1+\gamma_A)^t}$$

where γ_Y is the growth of output and γ_y is the growth of per-capita output.

After detrending and substituting out p_{it}, the first-order conditions can be written as follows:

$$\psi c_{it} l_{it} = (1-\alpha)(1-\phi)(1-l_{it}) y_{it} \tag{5.1}$$

$$1 = \frac{\beta}{1+\gamma_y} \frac{c_{it}}{c_{it+1}} \left[1 - \delta_k + \alpha(1-\phi) \frac{y_{i,t+1}}{k_{i,t+1}}\right] \tag{5.2}$$

$$1 = \frac{\beta}{1+\gamma_y} \frac{c_{it}}{c_{it+1}} \left[1 - \delta_m + \phi \left(\frac{N_{io} y_{i,t+1}}{\hat{m}_{i,t+1}} + \sum_{j \neq i} \omega_{jt+1} \frac{N_{jo} y_{jt+1}}{\hat{m}_{jt+1}} \right)\right] \tag{5.3}$$

$$y_{it} = a_{it}\hat{m}_{it}^{\phi}k_{it}^{\alpha(1-\phi)}l_{it}^{(1-\alpha)(1-\phi)} \tag{5.4}$$

$$\hat{m}_{it} = m_{it} + \omega_{it}m_{-it} \tag{5.5}$$

$$\begin{aligned} N_{i0}c_{it} + N_{i0}x_{ikt} + x_{imt} + \phi\omega_{it}N_{i0}y_{it}/\hat{m}_{it}\sum_{j\neq i} m_{jt} \\ = N_{i0}y_{it} + \phi m_{it}\sum_{j\neq i}\omega_{jt}N_{j0}y_{jt}/\hat{m}_{jt} \end{aligned} \tag{5.6}$$

$$x_{ikt} = (1+\gamma_Y)\,k_{i,t+1} - (1-\delta_k)\,k_{it} \tag{5.7}$$

$$x_{imt} = (1+\gamma_Y)\,m_{i,t+1} - (1-\delta_m)\,m_{it} \tag{5.8}$$

for $i = 1 \ldots I$ and $t = 0, \ldots T-1$. Equations (5.7) and (5.8) are replaced by $x_{ikt} = 0$ or $x_{imt} = 0$ if constraints on investment bind. If we include the terminal conditions, $k_{iT} = k_{iT-1}$ and $m_{iT} = m_{iT-1}$, and initial conditions for k_{i0} and m_{i0}, then the system of equations has $8 \times I \times T$ equations and $8 \times I \times T$ unknowns, $\{c_{it}$, l_{it}, y_{it}, x_{ikt}, x_{imt}, k_{it}, m_{it}, $\hat{m}_{it}\}$, $i = 1, \ldots I$, $t = 0, \ldots, T-1$.