APPENDIX: A Critique of Structural VARs Using Real Business Cycle Theory
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Revised May 2005

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1. Introduction

This appendix contains the proofs of Propositions 1-4 in the main text. I start by introducing the economic model and some preliminary algebra used in the proofs.

2. The Economic Model

2.1. Nomenclature

Below I will use the following notation for the model variables:

N: population ($N_t = (1+g_n)^t$)

c: per-capita consumption

x: per-capita investment

k: per-capita net capital stock (beginning of period t stock has subscript t)

l: per-capita labor input

tr: per-capita government transfers

G: total government spending

K: total stock of capital ($K_t = N_t k_t$)

L: total labor input in production ($L_t = N_t l_t$)

Z: labor-augmenting technical change ($Z_t = Z_{t-1} z_t$)

z: the innovation to technology

r: rental rate on capital

w: wage rate

τ_v: tax rate on v

$\hat{v}$: detrended, per-capita variable V ($\hat{v}_t = V_t/[N_t Z_t]$) with the exception of k

$\hat{k}$: detrended, per-capita capital, $\hat{k}_t = K_t/[N_t Z_{t-1}]$

2.2. Maximization problems

Consider an economy with households, firms, and the government. The representative household chooses consumption, investment, and labor to solve the following maximization problem:

$$
\begin{aligned}
&\max_{\{c_t,x_t,l_t\}} E\sum_{t=0}^{\infty}\beta^t\,U(c_t,1-l_t)N_t\\
\text{subject to}\quad &(1+\tau_{ct})c_t+(1+\tau_{xt})x_t=(1-\tau_{kt})r_tk_t+(1-\tau_{lt})w_tl_t+\tau_{kt}\delta k_t+tr_t\\
&N_{t+1}k_{t+1}=[(1-\delta)k_t+x_t]N_t\\
&c_t,x_t\geq 0\quad\text{in all states}
\end{aligned}
$$

taking processes for the rental rate, wage rate, the tax rates, and transfers as given. The representative firm solves a simple static problem at t:

$$\max_{\{K_t,L_t\}}\; F(K_t,Z_tL_t)-r_tK_t-w_tL_t.$$

The government sets rates of taxes and transfers in such a way that their budget constraint at t, namely,

$$G_t+N_ttr_t=\tau_{kt}(r_t-\delta)N_tk_t+\tau_{lt}w_tl_tN_t+\tau_{ct}N_tc_t+\tau_{xt}N_tx_t$$

is satisfied. In equilibrium, the following conditions must hold:

$$
\begin{aligned}
N_t(c_t+x_t)+G_t&=F(K_t,Z_tL_t) &(2.1)\\
N_tk_t&=K_t\\
N_tl_t&=L_t.
\end{aligned}
$$

2.3. First-order conditions

I now derive first-order conditions in this economy. The Lagrangian for the household optimization problem is given by

$$
\begin{aligned}
\mathcal{L}=E\sum_t\beta^tN_t\Big\{&U(c_t,1-l_t)\\
&+\mu_t\Big\{(1-\tau_{kt})r_tk_t+(1-\tau_{lt})w_tl_t+\tau_{kt}\delta k_t+tr_t-(1+\tau_{ct})c_t-(1+\tau_{xt})x_t\Big\}\\
&+\lambda_t\Big\{(1-\delta)k_t+x_t-(1+g_n)k_{t+1}\Big\}\Big\}
\end{aligned}
$$

Here, I am assuming that the investment decision will be interior.

The relevant first-order conditions are found by taking derivatives of $\mathcal{L}$ with respect to c_t, l_t, x_t, and k_{t+1}:

$$
\begin{aligned}
0 &= U_1(c_t, 1-l_t) - \mu_t(1+\tau_{ct}) \\
0 &= -U_2(c_t, 1-l_t) + \mu_t(1-\tau_{lt})w_t \\
0 &= \mu_t(1+\tau_{xt}) + \lambda_t = 0 \\
0 &= -(1+g_n)\lambda_t + E_t\{\mu_{t+1}[(1-\tau_{kt+1})r_{t+1} + \delta\tau_{kt+1}] + \lambda_{t+1}(1-\delta)\}
\end{aligned}
$$

Eliminating multipliers yields:

$$\frac{U_2(c_t, 1-l_t)}{U_1(c_t, 1-l_t)} = \frac{1-\tau_{lt}}{1+\tau_{ct}} w_t \tag{2.2}$$

$$\frac{1+\tau_{xt}}{1+\tau_{ct}} U_1(c_t, 1-l_t) = \beta E_t\left[\frac{U_1(c_{t+1}, 1-l_{t+1})}{1+\tau_{ct+1}}\left\{(1-\tau_{kt+1})r_{t+1} + \delta\tau_{kt+1} + (1-\delta)(1+\tau_{xt+1})\right\}\right]. \tag{2.3}$$

In addition, there are first-order conditions for the firm's static problem. These are

$$r_t = F_1(K_t, Z_t L_t) \tag{2.4}$$

$$w_t = F_2(K_t, Z_t L_t) Z_t. \tag{2.5}$$

Finally, I have a resource constraint given by (2.1).

From here on, I make the following functional form assumptions and auxiliary choices:

$$F(k, l) = k^\theta l^{1-\theta} \tag{2.6}$$

$$U(c, 1-l) = (c(1-l)^\psi)^{1-\sigma}/(1-\sigma) \tag{2.7}$$

$$\tau_{kt} = \tau_{ct} = 0$$

$$s_t = [\log z_t, \tau_{lt}, \tau_{xt}, \log \hat{g}_t]'$$

$$s_{t+1} = P_0 + P s_t + Q\eta_{s,t+1}, \quad \eta_s \sim N(0_{4\times 1}, I_{4\times 4}). \tag{2.8}$$

I have turned off τ_c since it plays a similar role to τ_n in distorting the labor-leisure choice. Similarly, I have turned off τ_k since it plays a similar role to τ_x in distorting the intertemporal margin.

If I substitute the choices (2.6)-(2.7) into (2.1) and (2.2)-(2.5), then substitute the equilibrium rates r_t and w_t into (2.2) and (2.3), I have:

$$N_t(c_t + g_t) + N_{t+1}k_{t+1} - (1-\delta)N_t k_t = (N_t k_t)^\theta (Z_t N_t l_t)^{1-\theta} \tag{2.9}$$

$$\frac{\psi c_t}{1-l_t} = (1-\tau_{lt})(1-\theta)(N_t k_t)^\theta Z_t^{1-\theta}(N_t l_t)^{-\theta} \tag{2.10}$$

$$\begin{aligned}(1+\tau_{xt})c_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ &= \beta E_t\big[c_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}\big\{\theta(N_{t+1}k_{t+1})^{\theta-1}(Z_{t+1}N_{t+1}l_{t+1})^{1-\theta} \\ &\qquad + (1-\delta)(1+\tau_{xt+1})\big\}\big].\end{aligned} \tag{2.11}$$

2.4. Log-linear computation

I first normalize the variables as follows:

$$\hat{c}_t = c_t/Z_t, \hat{x}_t = x_t/Z_t, \hat{g}_t = g_t/Z_t, \hat{y}_t = y_t/Z_t, \hat{k}_t = k_t/Z_{t-1}.$$

Using the functional forms for F and U in (2.6) and (2.7), respectively, the equilibrium rental and wage rates are:

$$\begin{aligned} r_t &= \theta K_t^{\theta-1}(Z_t L_t)^{1-\theta} = \theta \hat{k}_t^{\theta-1}(z_t l_t)^{1-\theta} \\ w_t &= (1-\theta)K_t^\theta (Z_t L_t)^{-\theta} Z_t = (1-\theta)\hat{k}_t^\theta (z_t l_t)^{-\theta} Z_t. \end{aligned}$$

This implies the following first-order conditions

$$\hat{c}_t + \hat{g}_t + (1+g_n)\hat{k}_{t+1} - (1-\delta)z_t^{-1}\hat{k}_t = \hat{y}_t = \hat{k}_t^\theta l_t^{1-\theta} z_t^{-\theta} \tag{2.12}$$

$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\theta)\hat{k}_t^\theta (z_t l_t)^{-\theta} \tag{2.13}$$

$$\begin{aligned}(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ &= \beta z_{t+1}^{-\sigma} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[\theta \hat{k}_{t+1}^{\theta-1}(z_{t+1}l_{t+1})^{1-\theta} + (1-\delta)(1+\tau_{xt+1})].\end{aligned} \tag{2.14}$$

Next, I compute the steady state of the system for constant values for z, the taxes, and government spending:

$$\begin{aligned} \hat{k}/l &= \left(\frac{(1+\tau_x)(1-\beta z^{-\sigma}(1-\delta))}{\beta z^{-\sigma}\theta z^{1-\theta}}\right)^{1/(\theta-1)} \\ \hat{c} &= \left[(\hat{k}/l)^{\theta-1}z^{-\theta} - (1+g_n) + (1-\delta)z^{-1}\right]\hat{k} - \hat{g} = \xi_1\hat{k} - \hat{g} \\ \hat{c} &= \left[(1-\tau_l)(1-\theta)(\hat{k}/l)^\theta z^{-\theta}/\psi\right](1 - 1/(\hat{k}/l)\ \hat{k}) = \xi_2 - \xi_3\hat{k} \end{aligned}$$

where the last 2 equations imply $\hat{k} = (\xi_2 + \hat{g})/(\xi_1 + \xi_3)$, $\hat{c} = \xi_1\hat{k} - \hat{g}$, $l = (1/(\hat{k}/l))\hat{k}$.

Assume that the solution for the capital decision takes the form:

$$\log \hat{k}_{t+1} = \gamma_k \log \hat{k}_t + \gamma \, [\, \log z_t \quad \tau_{lt} \quad \tau_{xt} \quad \log \hat{g}_t \,]' + \text{constant}, \tag{2.15}$$

where γ_k is a scalar and γ is 1×4 and equal to $[\gamma_z, \gamma_l, \gamma_x, \gamma_g]$. Assume the residual from the dynamic first-order condition (2.14) can be written (after substitutions from (2.12) and (2.13)):

$$\begin{aligned} & f(E_t \log \hat{k}_{t+2}, \log \hat{k}_{t+1}, \log \hat{k}_t, \log z_{t+1}, \log z_t, \tau_{lt+1}, \tau_{lt}, \tau_{xt+1}, \tau_{xt}, \log \hat{g}_{t+1}, \log \hat{g}_t) \\ & \quad \approx a_0 E_t \log \hat{k}_{t+2} + a_1 \log \hat{k}_{t+1} + a_2 \log \hat{k}_t + b_0 E_t s_{t+1} + b_1 s_t. \end{aligned}$$

Then the general solution algorithm is to find γ_k that solves the quadratic equation

$$a_0 \gamma_k^2 + a_1 \gamma_k + a_2 = 0,$$

and γ that solves the linear equations:

$$a_0 \gamma_k \gamma + a_0 \gamma P + a_1 \gamma + b_0 P + b_1 = 0_{1 \times 4}.$$

Note that this implies:

$$\gamma = -[(a_0 a + a_1) I_{4 \times 4} + a_0 P']^{-1} (b_0 P + b_1 I_{4 \times 4})'. \tag{2.16}$$

Once I have values for the the coefficients γ_k and γ, I can use (2.12) and (2.13) to back out c_t and l_t (either nonlinearly or by way of a log-linear approximation).

One property of the solution that I use later is the fact that $\gamma_k = -\gamma_z$. A second look at This is true because $\hat{k}_t$ is everywhere divided by z_t in the first-order conditions (2.12)-(2.13). Thus, when the first-order conditions are log-linearized, the same coefficients hit $\log(\hat{k}_t)$ and $-\log(z_t)$.

Given values for the coefficients in (2.15), I can derive expressions for labor, consumption, and investment using the static first-order conditions. In particular, I log-linearize (2.13) after substituting in for consumption from (2.12):

$$\begin{aligned} 0 \approx \psi \big\{ & \hat{k}^{\theta} l^{1-\theta} z^{-\theta} [\theta (\log \hat{k}_t - \log z_t) + (1-\theta) \log l_t] \\ & - (1+g_n) \hat{k} \log \hat{k}_{t+1} + (1-\delta) z^{-1} \hat{k} (\log \hat{k}_t - \log z_t) - \hat{g} \log \hat{g}_t \big\} \\ & + (1-\theta)(1-\tau_l) \hat{k}^{\theta} (zl)^{-\theta} (1-l) \big\{ 1/(1-\tau_l) \, \tau_{lt} \\ & - \theta \log \hat{k}_t + \theta (\log l_t + \log z_t) + l/(1-l) \log l_t \big\}. \end{aligned}$$

which I write succinctly as

$$\log l_t = \phi_{lk} \log \hat{k}_t + \phi_{lz} \log z_t + \phi_{ll} \tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}.$$

With this equation for $\log l$, I use the production relation and the capital accumulation equation to write $\log \hat{y}$ and $\log \hat{x}$ as follows:

$$\begin{aligned}
\log \hat{y}_t &= (\theta + (1-\theta)\phi_{lk}) \log \hat{k}_t + ((1-\theta)\phi_{lz} - \theta) \log z_t \\
&\qquad + (1-\theta)[\phi_{ll}\tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}] \\
&\equiv \phi_{yk} \log \hat{k}_t + \phi_{yz} \log z_t + \phi_{yl}\tau_{lt} + \phi_{yg} \log \hat{g}_t + \phi_{yk'} \log \hat{k}_{t+1} \qquad (2.17) \\
\log \hat{x}_t &\approx (1+g_n)\hat{k}/\hat{x} \log \hat{k}_{t+1} - (1-\delta)z^{-1}\hat{k}/\hat{x}(\log \hat{k}_t - \log z_t) \\
&\equiv \phi_{xk} \log \hat{k}_t + \phi_{xz} \log z_t + \phi_{xk'} \log \hat{k}_{t+1}. \qquad (2.18)
\end{aligned}$$

Finally, I can log-linearize (2.12) to get

$$\begin{aligned}
\log \hat{c}_t &\approx \{\hat{y}[\theta(\log \hat{k}_t - \log z_t) + (1-\theta) \log l_t] - \hat{g} \log \hat{g}_t \\
&\quad - (1+g_n)\hat{k} \log \hat{k}_{t+1} + (1-\delta)z^{-1}\hat{k}[\log \hat{k}_t - \log z_t]\}/\hat{c} \\
&= [\theta\hat{y}/\hat{c} + (1-\theta)\phi_{lk}\hat{y}/\hat{c} + (1-\delta)\hat{k}/(\hat{c}z)] \log \hat{k}_t \\
&\quad - [\theta\hat{y}/\hat{c} - (1-\theta)\phi_{lz}\hat{y}/\hat{c} + (1-\delta)\hat{k}/(\hat{c}z)] \log z_t \\
&\quad + [(1-\theta)\phi_{ll}\hat{y}/\hat{c}]\tau_{lt} \\
&\quad + [(1-\theta)\phi_{lg}\hat{y}/\hat{c} - \hat{g}/\hat{c}] \log \hat{g}_t \\
&\quad + [(1-\theta)\phi_{lk'}\hat{y}/\hat{c} - (1+g_n)\hat{k}/\hat{c}] \log \hat{k}_{t+1} \\
&\equiv \phi_{ck} \log \hat{k}_t + \phi_{cz} \log z_t + \phi_{cl}\tau_{lt} + \phi_{cg} \log \hat{g}_t + \phi_{ck'} \log \hat{k}_{t+1}. \qquad (2.19)
\end{aligned}$$

3. VARs and the 2-Shock Version of the Model

3.1. The Decision Functions

Assume the economy has only two shocks and they are orthogonal: a unit root in technology $\log z$ and an AR(1) in the tax rate on labor τ. (For convenience I drop l on τ_{lt} throughout this section.) The capital decision function has the form:

$$\log \hat{k}_{t+1} = \gamma_0 + \gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_t$$

and the labor decision function can be written:

$$\begin{aligned}
\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk} \log \hat{k}_t + \phi_{lk'} \log \hat{k}_{t+1} \\
&= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk} \log \hat{k}_t + \phi_{lk'}[\gamma_0\gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l\tau_t] \\
&= (\phi_{lk} + \phi_{lk'}\gamma_k) \log \hat{k}_t + (\phi_{lz} + \phi_{lk'}\gamma_z) \log z_t + (\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t.
\end{aligned}$$

These imply that output from a Cobb-Douglas production technology with capital share θ is:

$$\begin{aligned}
\log \hat{y}_t &= \theta(\log \hat{k}_t - \log z_t) + (1-\theta)\log l_t \\
&= (\theta + (1-\theta)\phi_{lk})\log \hat{k}_t - (\theta - (1-\theta)\phi_{lz})\log z_t + (1-\theta)\phi_{ll}\tau_t \\
&\qquad + (1-\theta)\phi_{lk'}\log \hat{k}_{t+1} \\
&= (\theta + (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k))\log \hat{k}_t - (\theta - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z))\log z_t \\
&\qquad + (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t
\end{aligned}$$

I can write the capital stock in terms of all lagged shocks as follows:

$$\begin{aligned}
\log \hat{k}_t &= \gamma_0 + \gamma_k(\gamma_0 + \gamma_k \log \hat{k}_{t-2} + \gamma_z \log z_{t-2} + \gamma_l \tau_{t-2}) + \gamma_z \log z_{t-1} + \gamma_l \tau_{t-1} \\
&= \gamma_0[1 + \gamma_k + \gamma_k^2 + \ldots] \\
&\quad + \gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\quad + \gamma_l[\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots]
\end{aligned}$$

or in differences as follows:

$$\begin{aligned}
\log \hat{k}_t - \log \hat{k}_{t-1} = \gamma_z[&\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&+ \gamma_l[\tau_{t-1} + (\gamma_k - 1)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}]
\end{aligned}$$

or in quasi-differences as follows:

$$\begin{aligned}
\log \hat{k}_t - \alpha \log \hat{k}_{t-1} = \gamma_z[&\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&+ \gamma_l[\tau_{t-1} + (\gamma_k - \alpha)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}]
\end{aligned}$$

I can also write hours in terms of past shocks as follows:

$$\begin{aligned}
\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk}\log \hat{k}_t + \phi_{lk'}\log \hat{k}_{t+1} \\
&= \phi_{lz}\log z_t + \phi_{ll}\tau_t \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\qquad + \phi_{lk}\gamma_l[\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots] \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + \gamma_k \log z_{t-1} + \gamma_k^2 \log z_{t-2} + \ldots] \\
&\qquad + \phi_{lk'}\gamma_l[\tau_t + \gamma_k \tau_{t-1} + \gamma_k^2 \tau_{t-2} + \ldots] \\
&= [(\phi_{lz} + \phi_{lk'}\gamma_z)\log z_t + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_z \log z_{t-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_z \log z_{t-2} + \ldots] \\
&\quad + [(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_l\tau_{t-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_l\tau_{t-2} + \ldots]
\end{aligned}$$

where I have ignored constant terms.

I can write logged hours in differences as follows:

$$
\begin{aligned}
\log l_t - \log l_{t-1} &= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_t - \tau_{t-1}) \\
&\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \log \hat{k}_{t-1}) \\
&= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_t - \tau_{t-1}) \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - 1)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_l[\tau_t + (\gamma_k - 1)\{\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_l[\tau_{t-1} + (\gamma_k - 1)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}] \\
&= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)] \log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\
&\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}
\end{aligned}
$$

or in quasi-difference form as follows:

$$
\begin{aligned}
\log l_t - \alpha \log l_{t-1} &= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_t - \alpha \tau_{t-1}) \\
&\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \alpha \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \alpha \log \hat{k}_{t-1}) \\
&= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_t - \alpha \tau_{t-1}) \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - \alpha)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_l[\tau_t + (\gamma_k - \alpha)\{\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_l[\tau_{t-1} + (\gamma_k - \alpha)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}] \\
&= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\alpha\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - \alpha)] \log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\alpha\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - \alpha)]\tau_{t-1} \\
&\quad + \gamma_l(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}
\end{aligned}
$$

I can use the expressions for output and hours to write out the change in productivity as follows:

$$
\begin{aligned}
\log(y_t/l_t) - \log(y_{t-1}/l_{t-1})& \\
&= \log \hat{y}_t - \log \hat{y}_{t-1} + \log z_t - \log l_t - \log l_{t-1} \\
&= \log z_t + \theta(\log \hat{k}_t - \log \hat{k}_{t-1} - \log l_t + \log l_{t-1} - \log z_t + \log z_{t-1}) \\
&= (1-\theta)\log z_t + \theta \log z_{t-1} \\
&\quad - \theta(\log l_t - \log l_{t-1} - \log \hat{k}_t + \log \hat{k}_{t-1}) \\
&= (1-\theta)\log z_t + \theta \log z_{t-1} - \theta\{[\phi_{lz} + \phi_{lk'}\gamma_z]\log z_t \\
&\quad - [\phi_{lz} - (\phi_{lk} - 1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)]\log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\phi_{ll} - (\phi_{lk} - 1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\
&\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots]\} \\
&= \{1 - \theta - \theta[\phi_{lz} + \phi_{lk'}\gamma_z]\}\log z_t \\
&\quad + \theta[1 + \phi_{lz} - (\phi_{lk} - 1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)]\log z_{t-1} \\
&\quad - \theta\gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\quad - \theta[\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t \\
&\quad + \theta[\phi_{ll} - (\phi_{lk} - 1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\
&\quad - \theta\gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2\tau_{t-4} + \ldots]
\end{aligned}
$$

3.2. The Model's Moving Average

The moving average for the model is given by:

$$
\begin{bmatrix} (1-L)\log y_t/l_t \\ (1-\alpha L)\log l_t \end{bmatrix} \equiv X_t = D_0\omega_t + D_1\omega_{t-1} + D_2\omega_{t-2} + \ldots
$$

where $\omega_t = [\log z_t, \tau_t]'$ and

$$
\begin{aligned}
D_0 &= \begin{bmatrix} 1 - \theta - \theta(\phi_{lz} + \phi_{lk'}\gamma_z) & -\theta(\phi_{ll} + \phi_{lk'}\gamma_l) \\ \phi_{lz} + \phi_{lk'}\gamma_z & \phi_{ll} + \phi_{lk'}\gamma_l \end{bmatrix} \\
D_1 &= \begin{bmatrix} \theta(1 + \phi_{lz} - (\phi_{lk} - 1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)) & \theta(\phi_{ll} - (\phi_{lk} - 1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)) \\ -\alpha\phi_{lz} + (\phi_{lk} + \phi_{lk'}(\gamma_k - \alpha))\gamma_z & -\alpha\phi_{ll} + (\phi_{lk} + \phi_{lk'}(\gamma_k - \alpha))\gamma_l \end{bmatrix} \\
D_2 &= \begin{bmatrix} -\theta\gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1] & -\theta\gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1] \\ (\phi_{lk} + \phi_{lk'}\gamma_k)(\gamma_k - \alpha)\gamma_z & (\phi_{lk} + \phi_{lk'}\gamma_k)(\gamma_k - \alpha)\gamma_l \end{bmatrix}
\end{aligned}
$$

and $D_j = \gamma_k D_{j-1}$ for $j \geq 3$.

Let $a = \phi_{lk} + \phi_{lk'}\gamma_k$ and $b = \phi_{ll} + \phi_{lk'}\gamma_l$. Also, note that $\phi_{lz} = -\phi_{lk}$ and $\gamma_z = -\gamma_k$ hold in the model economy with a unit root in technology.

$$D_0 = \begin{bmatrix} 1-\theta+\theta a & -\theta b \\ -a & b \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \theta(1-\gamma_k)(1-a) & \theta(b+(1-a)\gamma_l) \\ (\alpha-\gamma_k)a & -\alpha b+\gamma_l a \end{bmatrix}$$

$$D_2 = \begin{bmatrix} \gamma_k(1-a)\theta(1-\gamma_k) & -\gamma_l(1-a)\theta(1-\gamma_k) \\ \gamma_k a(\alpha-\gamma_k) & -\gamma_l a(\alpha-\gamma_k) \end{bmatrix}$$

and, again, $D_j = \gamma_k^{j-2} D_2$ for $j \geq 3$. Note that D_2 is singular.

If τ_t is an AR(1), it is more convenient to write the MA process in terms of $\eta_t = [\log z_t, \eta_{lt}]$ rather than in terms of ω_t. In this case,

$$X_t = D_0\eta_t + (D_0P+D_1)\eta_{t-1} + (D_0P^2+D_1P+D_2)\eta_{t-2} + (D_0P^3+D_1P^2+D_2P+D_3)\eta_{t-3}\ldots.$$

I normalize the MA so it has an identity for the first coefficient. That is, I set $C_0 = I$, $C_1 = (D_0P + D_1)D_0^{-1}$, and $C_j = C_{j-1}D_0PD_0^{-1} + D_jD_0^{-1}$.

3.3. Special Property of the D's

Next, I will show that the D matrices have a special property that will be exploited when I characterize coefficients of the VAR found by regressing X_t on lags of itself. The D's for the RBC model satisfy the relation:

$$\left(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1}\right) D_2 = 0. \tag{3.1}$$

One method of proof is to multiply all terms of the matrices in (3.1) and show that all elements are zero. I have done this but the algebra is messy.

A simpler proof is as follows. Note that

$$D_2 = \begin{bmatrix} (1-a)\theta(1-\gamma_k) \\ (\alpha-\gamma_k)a \end{bmatrix} \begin{bmatrix} \gamma_k & -\gamma_l \end{bmatrix} \equiv gh'. \tag{3.2}$$

Thus, I can rewrite the left hand side of (3.1) as follows

$$\begin{aligned} &\left(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1}\right)D_2 \\ &\quad = \left[\gamma_k(gh') - (gh')(D_0P + D_1)^{-1}(gh')\right] - \left[(D_0P + D_1)P(D_0P + D_1)^{-1}gh'\right]. \end{aligned} \tag{3.3}$$

I will prove that both terms in (3.3) in square brackets is equal to 2×2 zero matrices. The first step of the proof is to show that

$$(D_0 P + D_1)^{-1} g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{3.4}$$

The proof of this step is trivial since the first column of D_0P+D_1 is equal to g. Substituting (3.4) into (3.3), the result (3.1) follows immediately from the fact that $h'[1,0]' = \gamma_k$ and $P[1,0]' = 0$.

3.4. VAR Coefficients

Given expressions for the D coefficients in the model MA, and thus the normalized C coefficients, I can directly write out expressions for the coefficients in the VAR of X_t regressed on lags of itself. I will denote the VAR coefficients by B_j, $j = 1, 2, \ldots$. They are related to the MA coefficients as follows:

$$B_j = C_j - B_1 C_{j-1} - B_2 C_{j-2} - \ldots B_{j-1} C_1. \tag{3.5}$$

3.5. Proposition 1: Model has infinite-order VAR

Proposition 1. The model described above has a VAR representation with coefficients B_j that satisfy

$$B_j = M B_{j-1} \tag{3.6}$$

for $j \geq 2$, with $B_1 = C_1 = (D_0 P + D_1) D_0^{-1}$. The matrix M is a 2×2 matrix with eigenvalues equal to α and

$$\frac{\gamma_k - \gamma_l a/b - \theta}{1 - \theta},$$

where $a = \phi_{lk} + \phi_{lk'}\gamma_k$ and $b = \phi_{ll} + \phi_{lk'}\gamma_l$ are the coefficients on k and τ_l in the labor decision function.

Proof of Proposition 1. Choose $M = C_2 C_1^{-1} - C_1$. Using the formula (3.5) for the VAR coefficient, it is easy to show that $M = B_2 B_1^{-1}$. Therefore, $B_2 = M B_1$ holds. Consider the next coefficient. Using the formula (3.5), I have

$$\begin{aligned} B_3 - M B_2 &= C_3 - B_1 C_2 - B_2 C_1 - M(C_2 - B_1 C_1) \\ &= C_3 - B_1 C_2 - M C_2 \\ &= C_3 - C_1 C_2 - (C_2 C_1^{-1} - C_1) C_2 \end{aligned}$$

$$
\begin{aligned}
&= C_3 - C_2C_1^{-1}C_2 \\
&= C_2D_0PD_0^{-1} + \gamma_k D_2D_0^{-1} - C_2C_1^{-1}(C_1D_0PD_0^{-1} + D_2D_0^{-1}) \\
&= \gamma_k D_2D_0^{-1} - C_2C_1^{-1}D_2D_0^{-1}) \\
&= (\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= (\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1})D_2D_0^{-1} \\
&= 0
\end{aligned}
$$

where the last relation follows from intermediate calculations done in Section 3. The same calculation can be done for any j^* using the fact that (3.6) holds for all $j < j^*$, namely

$$
\begin{aligned}
B_j - MB_{j-1} &= C_j - B_1C_{j-1} \ldots - B_{j-1}C_1 - M(C_{j-1} - \ldots B_{j-2}C_1) \\
&= C_j - B_1C_{j-1} - MC_{j-1} \\
&= C_j - C_1C_{j-1} - (C_2C_1^{-1} - C_1)C_{j-1} \\
&= C_j - C_2C_1^{-1}C_{j-1} \\
&= C_{j-1}D_0PD_0^{-1} + D_jD_0^{-1} - C_2C_1^{-1}(C_{j-2}D_0PD_0^{-1} + D_{j-1}D_0^{-1}) \\
&= C_{j-1}D_0PD_0^{-1} + \gamma_k^{j-2}D_2D_0^{-1} - C_2C_1^{-1}(C_{j-2}D_0PD_0^{-1} + \gamma_k^{j-3}D_2D_0^{-1}) \\
&= (C_{j-1} - C_2C_1^{-1}C_{j-2})D_0PD_0^{-1} + \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= (B_{j-1} - MB_{j-2})D_0PD_0^{-1} + \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= \gamma_k^{j-3}(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1})D_2D_0^{-1} \\
&= 0.
\end{aligned}
$$

Next, I prove that the two eigenvalues of M are $\lambda_1 = \alpha$ and $\lambda_2 = (\gamma_k - \gamma_l a/b - \theta)/(1-\theta)$. One way to do this is to write out all of the terms for matrix M and derive expressions for the trace and the determinant. The trace is equal to the sum of the eigenvalues and the determinant is equal to the product of the eigenvalues. This is 2 equations and 2 unknowns. I have done this but the algebra is messy.

A simpler proof that the eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = (\gamma_k - \gamma_l a/b - \theta)/(1-\theta)$ is as follows. Using (3.2) and the definitions of the C's in terms of the D's, I can derive the following expression for M in terms of the D's, P, and h:

$$
\begin{aligned}
M &= C_2C_1^{-1} - C_1 \\
&= (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1} \\
&= (D_0P + D_1)P(D_0P + D_1)^{-1} + D_2(D_0P + D_1)^{-1} \\
&\quad - (D_0P + D_1)D_0^{-1}(D_0P + D_1)(D_0P + D_1)^{-1}
\end{aligned}
$$

$$= D_2(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1}D_1(D_0P + D_1)^{-1}$$
$$= (D_0P + D_1)[1,0]'h'(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1}D_1(D_0P + D_1)^{-1}$$
$$= (D_0P + D_1)([1,0]'h' - D_0^{-1}D_1)(D_0P + D_1)^{-1}.$$

Appealing to standard results in linear algebra, the eigenvalues of M are equal to the eigenvalues of the simpler matrix $[1,0]'h' - D_0^{-1}D_1$, which is equal to

$$[1,0]'h' - D_0^{-1}D_1 = \frac{1}{1-\theta}\begin{bmatrix} \gamma_k - \theta(1-a+a\alpha) & -\gamma_l + \theta b(1-\alpha) \\ (\gamma_k - \alpha - \theta(1-a)(1-\alpha))a/b & \alpha - \theta(a + \alpha(1-a)) - \gamma_l a/b \end{bmatrix}$$

Taking the trace, I get

$$\text{trace}([1,0]'h' - D_0^{-1}D_1) = \alpha + \frac{\gamma_k - \gamma_l a/b - \theta}{1-\theta}. \tag{3.7}$$

Taking the determinant, I get

$$\det([1,0]'h' - D_0^{-1}D_1) = \alpha \times \frac{\gamma_k - \gamma_l a/b - \theta}{1-\theta}. \tag{3.8}$$

The two equations (3.7) and (3.8) uniquely determine the two eigenvalues which are those proposed. ■

3.6. Blanchard-Quah Identification

I now consider the procedure of Blanchard and Quah (1989) when applied to data from the 2-shock version of the model described above.

Blanchard and Quah start, as I did, with a VAR

$$X_t = B_1X_{t-1} + B_2X_{t-2} + \ldots B_pX_{t-p} + v_t, \qquad Ev_tv_t = \Omega$$

which is estimated using time series $\{X_t\}$. As described above, this implies the MA

$$X_t = v_t + C_1v_{t-1} + C_2v_{t-2} + \ldots. \tag{3.9}$$

Some structure is needed to derive a "structural MA" with shocks that have economic interpretation. In this case, I will use the following notation for the structural MA:

$$X_t = A_0\epsilon_t + A_1\epsilon_{t-1} + A_2\epsilon_{t-2} + \ldots. \tag{3.10}$$

where $A_0\epsilon_t = v_t$ and $A_j = C_jA_0$.

Because I will impose restrictions on the sums of the A's and C's, I define

$$\begin{aligned}\bar{C} &= I + C_1 + C_2 + C_3 + \dots \\ \bar{A} &= A_0 + A_1 + A_2 + A_3 + \dots \\ &= \bar{C} A_0.\end{aligned}$$

Since $A_0 \epsilon_t = v_t$, it must be the case that $A_0 E \epsilon_t \epsilon_t A_0' = \Omega$. Blanchard and Quah assume that the elements of ϵ_t are orthogonal and demand shocks do not have a long-run effect on productivity. Without loss of generality, we can normalize the magnitude of the variances of the elements of ϵ_t and assume, therefore, that

$$\begin{aligned}&A_0 A_0' = \Omega \\ &\bar{C}(1,1)A_0(1,2) + \bar{C}(1,2)A_0(2,2) = 0 \qquad (3.11)\end{aligned}$$

which is four equations in the four unknown elements of A_0. Condition (3.11) ensures that the demand shock does not have a long-run effect on productivity. Writing out the system of 4 equations and 4 unknowns yields:

$$\begin{aligned}\omega_{11} &= A_0(1,1)^2 + A_0(1,2)^2 \\ \omega_{12} &= A_0(1,1)A_0(2,1) + A_0(1,2)A_0(2,2) \qquad (3.12) \\ \omega_{22} &= A_0(2,1)^2 + A_0(2,2)^2 \\ 0 &= \bar{C}(1,1)A_0(1,2) + \bar{C}(1,2)A_0(2,2)\end{aligned}$$

Eliminate $A_0(1,2)$ using the fact that $A_0(1,2) = -\bar{C}(1,2)A_0(2,2)/\bar{C}(1,1)$:

$$\begin{aligned}\omega_{11} &= A_0(1,1)^2 + f^2 A_0(2,2)^2 \\ \omega_{12} &= A_0(1,1)A_0(2,1) + f A_0(2,2)^2 \\ \omega_{22} &= A_0(2,1)^2 + A_0(2,2)^2\end{aligned}$$

where $f = -\bar{C}(1,2)/\bar{C}(1,1)$. Solve for $A_0(1,1)$ and $A_0(2,1)$:

$$A_0(1,1) = [\omega_{11} - f^2 A_0(2,2)^2]^{1/2} \qquad (3.13)$$

$$A_0(2,1) = [\omega_{22} - A_0(2,2)^2]^{1/2} \qquad (3.14)$$

and substitute to get:

$$\omega_{12} = f A_0(2,2)^2 + [\omega_{11} - f^2 A_0(2,2)^2]^{1/2}[\omega_{22} - A_0(2,2)^2]^{1/2}.$$

Let $\lambda = A_0(2,2)^2$ and the result is a quadratic in λ:

$$(\omega_{12} - f\lambda)^2 = (\omega_{11} - f^2\lambda)(\omega_{22} - \lambda)$$

which can be written out:

$$\omega_{12}^2 - 2f\lambda\omega_{12} + f^2\lambda^2 = \omega_{11}\omega_{22} - f^2\lambda\omega_{22} - \omega_{11}\lambda + f^2\lambda^2$$

and simplified as follows:

$$\lambda = \frac{\omega_{11}\omega_{22} - \omega_{12}^2}{\omega_{11} + f^2\omega_{22} - 2f\omega_{12}}. \tag{3.15}$$

In addition, I need to impose sign conventions since impulse responses can be either positive or negative. I will consider one sign convention for the demand shock and two different sign conventions for the technology shock.

The demand shock in our example is a shock to the tax rate on labor. For this choice of shock, I want to impose $A_0(2,2) < 0$ so that hours fall with a positive shock to the tax rate on labor. For $A_0(2,2)$, it must be the case that $A_0(2,2) = -\sqrt{\lambda}$ since λ is positive. Given $A_0(2,2)$, it immediately follows that $A_0(1,2) = fA_0(2,2)$.

3.6.1. Sign convention on $A_0(1,1)$

For the technology shock, I first consider the sign convention that productivity rises on impact in response to a positive technology shock, namely $A_0(1,1) > 0$. In this case, I need to use the positive root of $A_0(1,1)^2 = \omega_{11} - f^2\lambda$:

$$A_0(1,1) = \sqrt{\omega_{11} - f^2\lambda}.$$

Given a value for $A_0(1,1)$, I have $A_0(2,1)$ from:

$$A_0(2,1) = (\omega_{12} - f\lambda)/A_0(1,1).$$

3.6.2. Sign convention on $\bar{A}(1,1)$

The second sign convention assumes that productivity is positive in the long run so that $\bar{A}(1,1) > 0$ and therefore

$$\bar{C}(1,1)A_0(1,1) + \bar{C}(1,2)A_0(1,2) > 0.$$

This condition can also be written in terms of $A_0(1,1)$ and known parameters:

$$\bar{C}(1,1)A_0(1,1) + \bar{C}(1,2)(\omega_{1,2} - f\lambda)/A_0(1,1) > 0. \tag{3.16}$$

In this case, I choose the sign on the square root of $\omega_{11} - f^2\lambda$ so that (3.16) is satisfied.

3.6.3. Full solution

The full solution is

$$\begin{aligned}
A_0(2,2) &= -\sqrt{\lambda} \\
A_0(1,2) &= f A_0(2,2) \\
A_0(1,1) &= \text{root of } \omega_{11} - f^2\lambda \text{ satisfying sign convention} \\
A_0(2,1) &= (\omega_{12} - f\lambda)/A_0(1,1)
\end{aligned}$$

where λ is defined in (3.15) and $f = -\bar{C}(1,2)/\bar{C}(1,1)$.

3.6.4. Cholesky decomposition

In the literature, many report using the following formula for A_0:

$$A_0 = \bar{C}^{-1} L$$

where L is a lower triangular matrix such that with positive elements on the diagonal that satisfies $LL' = \bar{C}\Omega\bar{C}'$. This choice imposes the long-run restriction in (3.11) and the long-run sign convention $\bar{A}(1,1)$ automatically.

It does not impose $A_0(2,2) < 0$. However, in most cases, responses to demand shocks are not discussed.

3.7. Proposition 2: OLS Results

Proposition 1 says that the model has an infinite-lag vector autoregressive structure. The next proposition considers the outcome when OLS regressions are run with one lag. Let $V_0 = EX_tX_t'$ be the theoretical variance matrix for X_t. Let $V_1 = EX_tX_{t-1}'$ be the covariance matrix for X_t and its lag. If $ED_0\eta_t\eta_t'D_0' = \Omega$ is the theoretical variance-covariance of the model's shock vector, then

$$V_0 = \Omega + C_1\Omega C_1' + C_2\Omega C_2' + \dots \tag{3.17}$$

$$V_1 = C_1\Omega + C_2\Omega C_1' + C_3\Omega C_2' + \dots. \tag{3.18}$$

Proposition 2. Assume that a regression is run of the form

$$X_t = B_{ols}X_{t-1} + v_t, \quad Ev_tv_t' = \Omega_{ols}$$

with X_t from the RBC model. Then, the variance-covariance matrix is

$$\Omega_{ols} = V_0 - V_1 V_0^{-1} V_1' \tag{3.19}$$
$$= \Omega + M\Omega M' - M\Omega V_0^{-1}\Omega M' \tag{3.20}$$

where $M = C_2 C_1^{-1} - C_1$ and the inverse of the sum of MA coefficients is

$$\begin{aligned}\bar{C}_{ols}^{-1} &= I - B_{ols} \\ &= \bar{C}^{-1} + M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1}.\end{aligned}$$

In other words, the OLS matrices Ω_{ols} and $\bar{C}_{ols}$ are not equal to their theoretical counterparts, Ω and $\bar{C}$.

Proof of Proposition 2. The relation (3.19) follows from the standard projection formulas,

$$B_{ols} = (EX_t X_{t-1})(EX_{t-1}X_{t-1}')^{-1} = V_1 V_0^{-1}$$
$$Ev_t v_t' = E(X_t - B_{ols}X_{t-1})(X_t - B_{ols}X_{t-1})' = V_0 - V_1 V_0^{-1} V_1'. \tag{3.21}$$

Before substituting in (3.17) and (3.18), I can exploit the nature of the model's MA. In particular, I can use the fact that

$$C_j = (C_1 + M)C_{j-1}, \tag{3.22}$$

which follows from the formula (3.5) and Proposition 1. That is,

$$\begin{aligned}C_j - (C_1 + M)C_{j-1} &= (B_j + B_{j-1}C_1 + B_{j-2}C_2 + \ldots + B_1 C_{j-1}) \\ &\qquad - C_1 C_{j-1} - M(B_{j-1} + B_{j-2}C_1 + \ldots + B_1 C_{j-2}) \\ &= B_1 C_{j-1} - C_1 C_{j-1} \\ &= 0.\end{aligned}$$

Thus, I can write V_0 as follows:

$$V_0 = \Omega + C_1\Omega C_1' + (C_1 + M)C_1\Omega C_1'(C_1 + M)' + (C_1 + M)^2 C_1 \Omega C_1'(C_1 + M)^{2'} + \ldots$$

which implies that

$$V_0 = (C_1 + M)V_0(C_1 + M)' + \Omega + C_1\Omega C_1' - (C_1 + M)\Omega(C_1 + M)'. \tag{3.23}$$

For V_1,

$$V_1 = C_1\Omega + (C_1 + M)C_1\Omega C_1' + (C_1 + M)^2\Omega(C_1 + M) + \ldots$$
$$= (C_1 + M)V_0 - M\Omega. \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.21) yields

$$\begin{aligned}
V_0 - V_1 V_0^{-1} V_1' &= V_0 - [(C_1+M)V_0 - M\Omega]V_0^{-1}[(C_1+M)V_0 - M\Omega]' \\
&= V_0 - (C_1+M)V_0(C_1+M)' + M\Omega(C_1+M)' \\
&\qquad + (C_1+M)\Omega M' - M\Omega V_0^{-1}\Omega M' \\
&= \Omega + C_1\Omega C_1' - (C_1+M)\Omega(C_1+M)' + M\Omega(C_1+M)' \\
&\qquad + (C_1+M)\Omega M' - M\Omega V_0^{-1}\Omega M' \\
&= \Omega + M\Omega M' - M\Omega V_0^{-1}\Omega M'
\end{aligned}$$

which is the same as (3.20). This proves the first part of the proposition.

For the second part, I need to construct the matrix $\bar{C}_{ols}$ using the relation between the AR coefficients and the MA coefficients in (3.5). In this case,

$$\bar{C}_{ols} = (I - B_{ols})^{-1}.$$

Thus, I have

$$\begin{aligned}
\bar{C}_{ols} &= (I - V_1 V_0^{-1})^{-1} \\
&= (I - [(C_1+M)V_0 - M\Omega]V_0^{-1})^{-1} \\
&= (I - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (I - (I-M)^{-1}C_1 + (I-M)^{-1}C_1 - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (\bar{C}^{-1} + (I + M + M^2 + \ldots)C_1 - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (\bar{C}^{-1} + M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1})^{-1}. \qquad (3.25)
\end{aligned}$$

The term $M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1}$ is not generically zero. This proves the second part of the proposition. ■

The term $M\Omega M' - M\Omega V_0^{-1}\Omega M'$ is zero if the RBC model's VAR representation has only one lag (e.g., $M = 0$). It is close to zero if one of the shocks is close to 0. In this latter case, $\Omega V_0^{-1}\Omega \approx \Omega$ and the SVAR user detects correctly the variance of the one shock driving the system. This is true even if the VAR coefficients are wrong (e.g., M is very different than 0).

What happens if we have a VAR with n lags? In this case, the formula is messy but $Ev_t v_t'$ can be written

$$Ev_t v_t' = V_0 - [\,V_1 \quad V_2 \quad \cdots \quad V_n\,] \begin{bmatrix} V_0 & V_1 & \cdots & V_{n-1} \\ V_1' & V_0 & \cdots & V_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ V_{n-1}' & V_{n-2}' & \cdots & V_0 \end{bmatrix}^{-1} \begin{bmatrix} V_1' \\ V_2' \\ \vdots \\ V_n' \end{bmatrix}$$

with $V_j = (C_1 + M)^{j-1} V_1$, where V_0 is the matrix in (3.23) and V_1 is the matrix in (3.24).

3.8. The Propositions for Two Special Cases

In this section, I consider two special cases. The first has $\theta = 0$. The second has $\sigma_\tau = 0$. I show in these very special cases that the SVAR can uncover the true impulse response for hours in response to a technology shock even if only one lag is used in the VAR regression.

3.8.1. Proposition 3a: No capital in the model

Proposition 3a. Assume that θ is set to 0 in the RBC model. If a regression is run of the form

$$X_t = B_{ols} X_{t-1} + v_t$$

with X_t from the RBC model, then the Blanchard-Quah procedure recovers the true impulse response function for hours in response to technology, namely

$$A_j(2,1) = 0 \tag{3.26}$$

for all j.

Proof of Proposition 3a. It is important to note that C_1 is singular in this case. Thus, I can't write M as $C_2 C_1^{-1} - C_1$, but rather, I simply work with:

$$M = \begin{bmatrix} 0 & 0 \\ M(2,1) & \alpha \end{bmatrix}$$

for arbitrary $M(2,1)$ (which is what I would have in the limit as θ goes to 0). It is easy to show that $B_2 = C_2 - C_1^2 = MB_1$, $B_3 = C_3 - B_2 C_2 - B_1 C_1 = MB_2$ and so on. To prove the result in (3.26), I need to show that the errors that a SVAR user encounters in estimating $Ev_t v_t'$ and $\bar{C}$ do not affect the (2,1) elements of the A_j's. Starting with $Ev_t v_t'$:

$$\begin{aligned} Ev_t v_t' &= \Omega + M\Omega M' - M\Omega V_0^{-1} M' \\ &= \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_\tau^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \end{aligned}$$

where the second matrix is $M\Omega M' - M\Omega V_0^{-1} M'$ with a nonzero (2,2) element x. The value of x does not affect the result, so I don't need to specify it precisely. Next, I consider the $\bar{C}$ matrix:

$$\bar{C}_{ols}^{-1} = (\bar{C}^{-1} + M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1})^{-1}$$
$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha)/(1-\rho) \end{bmatrix}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right)^{-1}$$

where y is a nonzero term in the SVAR error. The magnitude of y does not affect the result. Notice that the (1,1) and (1,2) elements of $\bar{C}$ are correctly computed. Notice also that this implies $f = -\bar{C}(1,2)/\bar{C}(1,1) = 0$ and therefore $A_0(2,1) = 0$. For all higher terms, $A_j(2,1) = 0$ because $V_1 V_0^{-1}$ has zeros in the first column. ■

3.8.2. Proposition 3b: Only one shock

Proposition 3b. Assume that $\sigma_\tau = 0$ in the RBC model. If a regression is run of the form

$$X_t = B_{ols} X_{t-1} + v_t$$

with X_t from the RBC model, then the Blanchard-Quah procedure recovers the true impulse responses to technology, namely

$$A_j = D_j Q$$

for all j.

Proof of Proposition 3b. The first part of the proof is concerned with the impact coefficient A_0. I show that $Ev_t v_t' = \Omega$ if $\sigma_\tau = 0$, where Ω is the true variance-covariance matrix for the model. This is the main step in showing that the impact coefficient is correct. Then I show that the other coefficients can also be recovered by the SVAR. From Proposition 2, the following holds for the one-lag regression regardless of the size of the shocks:

$$Ev_t v_t' = \Omega + M\Omega M' - M\Omega V_0^{-1}\Omega M'.$$

I now show that $\Omega = \Omega V_0^{-1}\Omega$ if $\sigma_\tau = 0$, and therefore $Ev_t v_t' = \Omega$. I do this in three steps. First, I show that

$$\Omega = \Omega(1,1) \begin{bmatrix} 1 & \zeta \\ \zeta & \zeta^2 \end{bmatrix} \tag{3.27}$$

where $\zeta = -a/(1-\theta+\theta a)$. Second, I show that

$$\frac{1}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} V_0 = \Omega \tag{3.28}$$

where $\nu = -\theta(1-\gamma_k)(1-a)/[(\alpha-\gamma_k)a]$. Third, I show that (3.27) and (3.28) imply:

$$\Omega - \Omega V_0^{-1}\Omega = 0. \tag{3.29}$$

Writing out Ω yields

$$\begin{aligned}
\Omega &= D_0 Q Q' D_0' \\
&= \begin{bmatrix} 1-\theta+\theta a & -\theta b \\ -a & b \end{bmatrix} \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\theta+\theta a & -a \\ -\theta b & b \end{bmatrix} \\
&= \begin{bmatrix} (1-\theta+\theta a)^2 & -(1-\theta+\theta a)a \\ -(1-\theta+\theta a)a & a^2 \end{bmatrix} \sigma_z^2 \\
&= \Omega(1,1) \begin{bmatrix} 1 & -a/(1-\theta+\theta a) \\ -a/(1-\theta+\theta a) & a^2/(1-\theta+\theta a)^2 \end{bmatrix}
\end{aligned}$$

and (3.27) holds. Writing out V_0 in (3.28) yields

$$\frac{1}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \left(\Omega + C_1 \Omega C_1' + C_2 \Omega C_2' + C_3 \Omega C_3' \ldots\right). \tag{3.30}$$

The second term on the right hand side of (3.30) is equal to a 2×2 matrix of zeros:

$$\begin{aligned}
&\begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} C_1 \Omega C_1' \\
&\qquad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \begin{bmatrix} \theta^2(1-\gamma_k)^2(1-a)^2 & \theta(1-\gamma_k)(1-a)(\alpha-\gamma_k)a \\ \theta(1-\gamma_k)(1-a)(\alpha-\gamma_k)a & (\alpha-\gamma_k)^2 a^2 \end{bmatrix} \sigma_z^2 \\
&\qquad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

All higher terms are also equal to 2×2 matrices of zeros:

$$\begin{aligned}
&\begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} C_j \Omega C_j' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (C_{j-1} D_0 P D_0^{-1} + D_j D_0^{-1})(D_0 Q Q' D_0')(C_{j-1} D_0 P D_0^{-1} + D_j D_0^{-1})' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (\gamma_k^{j-2} D_2)(QQ')(\gamma_k^{j-2} D_2)' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \gamma_k^{2j-4} \begin{bmatrix} [\gamma_k(1-a)\theta(1-\gamma_k)]^2 & \gamma_k^2(1-a)\theta(1-\gamma_k)a(\alpha-\gamma_k) \\ \gamma_k^2(1-a)\theta(1-\gamma_k)a(\alpha-\gamma_k) & [\gamma_k a(\alpha-\gamma_k)]^2 \end{bmatrix} \sigma_z^2 \\
&\quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{1+\zeta\nu}\begin{bmatrix}1 & \nu \\ \zeta & \zeta\nu\end{bmatrix}&(\Omega + C_1\Omega C_1' + C_2\Omega C_2' + C_3\Omega C_3' \ldots) \\ &= \frac{1}{1+\zeta\nu}\begin{bmatrix}1 & \nu \\ \zeta & \zeta\nu\end{bmatrix}\Omega \\ &= \frac{\Omega(1,1)}{1+\zeta\nu}\begin{bmatrix}1 & \nu \\ \zeta & \zeta\nu\end{bmatrix}\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} \\ &= \Omega\end{aligned}$$

which proves (3.28). Now, I am ready for

$$\begin{aligned}\Omega - \Omega V_0^{-1}\Omega &= \Omega(1,1)\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} - \frac{1}{1+\zeta\nu}\begin{bmatrix}1 & \nu \\ \zeta & \zeta\nu\end{bmatrix}\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} \\ &= \Omega(1,1)\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} - \frac{1}{1+\zeta\nu}\begin{bmatrix}1+\zeta\nu & \zeta(1+\zeta\nu) \\ \zeta(1+\zeta\nu) & \zeta^2(1+\zeta\nu)\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}\end{aligned}$$

which proves that there is no error in computing Ev_tv_t', that is $Ev_tv_t' = \Omega$. The next step is to show that this is all that is needed for the correct inference. Recall the formulas for the elements of A_0 and λ in Section 3.6. Because $\Omega_{ols} = \Omega$ and det(Ω)=0, it must be the case that $\lambda = 0$. Thus, $A_0(2,1)$ found by the SVAR is

$$A_0 = \begin{bmatrix}\sqrt{\omega_{11}} & 0 \\ \sqrt{\omega_{22}} & 0\end{bmatrix}$$

where $\sqrt{\omega_{jj}} = \sqrt{\Omega(j,j)}$. Using the formulas above, we have $A_0(1,1) = (1-\theta+\theta a)\sigma_z$ and $A_0(2,1) = -a\sigma_z$. Thus, $A_0 = D_0Q$. This proves that there is no mistaken inference for the impact coefficient. Next, I check A_j for $j > 1$, which is equal to $B_{ols}^jA_0$. For $j = 1$,

$$\begin{aligned}B_{ols}A_0 &= V_1V_0^{-1}A_0 \\ &= V_1V_0^{-1}D_0Q \\ &= (C_1 + M - M\Omega V_0^{-1})D_0Q \\ &= C_1D_0Q \\ &= (D_0P + D_1)D_0^{-1}D_0Q \\ &= D_1Q\end{aligned}$$

where I have used the fact that $(I - \Omega V_0^{-1})D_0Q = 0$. For $j = 2$,

$$B_{ols}^2A_0 = (V_1V_0^{-1})^2A_0$$

$$
\begin{aligned}
&= (V_1 V_0^{-1}) C_1 D_0 Q \\
&= (C_1 + M - M\Omega V_0^{-1}) C_1 D_0 Q \\
&= C_2 D_0 Q - M\Omega V_0^{-1} C_1 D_0 Q \\
&= C_2 D_0 Q \\
&= (D_0 P^2 + D_1 P + D_2) D_0^{-1} D_0 Q \\
&= D_2 Q
\end{aligned}
$$

where I have used the fact that $\Omega V_0^{-1} C_1 D_0 Q = 0$. Similarly, I can prove it for higher terms by noting that if $B_{ols}^{j-1} A_0 = C_{j-1} D_0 Q$ holds, then $B_{ols}^{j} A_0 = C_j D_0 Q$ holds and so does the following:

$$
\begin{aligned}
B_{ols}^{j} A_0 &= (V_1 V_0^{-1}) C_{j-1} D_0 Q \\
&= (C_1 + M - M\Omega V_0^{-1}) C_{j-1} D_0 Q \\
&= C_j D_0 Q - M\Omega V_0^{-1} C_{j-1} D_0 Q \\
&= C_j D_0 Q \\
&= (D_0 P^j + D_1 P + \ldots + D_j) D_0^{-1} D_0 Q \\
&= D_j Q.
\end{aligned}
$$

This establishes that in the case with $\sigma_\tau = 0$, the SVAR uncovers the true impulse responses to technology. ■

What is interesting about the last two propositions is that the special cases are not relevant for modern business cycle theorists. Modern business cycle theorists assume that both capital accumulation and shocks in addition to technology (e.g., distortions to labor) are quantitatively important. Furthermore, adding these factors is not a recent phenomena. They are central to the work following Kydland and Prescott (1982) (which includes my thesis).

4. VARs and 3-Shock Versions of the Model

I consider several versions of an RBC model with three shocks and three variables in the VAR. The first has a investment tax shock and the log of the investment-output ratio in the VAR. The second has a government spending shock and the log of the investment-output ratio in the VAR. The third has a investment tax shock and the log of the consumption-output ratio in the VAR.

4.1. Investment Tax Shock and Investment-Output in VAR

Assume the economy is an RBC model with three orthogonal shocks: a unit root in technology $\log z$, an AR(1) in the tax rate on labor τ_l, and an AR(1) in the tax rate on investment τ_x. The capital decision function has the form:

$$\log \hat{k}_{t+1} = \gamma_0 + \gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_{lt} + \gamma_x \tau_{xt} \tag{4.1}$$

and the labor decision function can be written:

$$\begin{aligned}
\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_{lt} + \phi_{lx}\tau_{xt} + \phi_{lk} \log \hat{k}_t + \phi_{lk'} \log \hat{k}_{t+1} \\
&= \phi_{lz} \log z_t + \phi_{ll}\tau_{lt} + \phi_{lx}\tau_{xt} + \phi_{lk} \log \hat{k}_t + \phi_{lk'}[\gamma_0\gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l\tau_{lt} + \gamma_x\tau_{xt}] \\
&= (\phi_{lk} + \phi_{lk'}\gamma_k) \log \hat{k}_t + (\phi_{lz} + \phi_{lk'}\gamma_z) \log z_t + (\phi_{ll} + \phi_{lk'}\gamma_l)\tau_{lt} + (\phi_{lx} + \phi_{lk'}\gamma_x)\tau_{xt}.
\end{aligned}$$

Note that I include the term $\phi_{lx}\tau_{xt}$ here even though it is equal to 0 in equilibrium. I do so because the same mathematics will be used later in the case of the government spending shock.

Next, I write out output from a Cobb-Douglas production technology with capital share θ is:

$$\begin{aligned}
\log \hat{y}_t &= \theta(\log \hat{k}_t - \log z_t) + (1-\theta) \log l_t \\
&= (\theta + (1-\theta)\phi_{lk}) \log \hat{k}_t - (\theta - (1-\theta)\phi_{lz}) \log z_t + (1-\theta)\phi_{ll}\tau_{lt} \\
&\qquad + (1-\theta)\phi_{lx}\tau_{xt} + (1-\theta)\phi_{lk'} \log \hat{k}_{t+1} \\
&= (\theta + (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k)) \log \hat{k}_t - (\theta - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z)) \log z_t \\
&\qquad + (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_{lt} \\
&\qquad + (1-\theta)(\phi_{lx} + \phi_{lk'}\gamma_x)\tau_{xt}
\end{aligned}$$

I can write the capital stock in terms of all lagged shocks as follows:

$$\begin{aligned}
\log \hat{k}_t &= \gamma_0 + \gamma_k(\gamma_0 + \gamma_k \log \hat{k}_{t-2} + \gamma_z \log z_{t-2} + \gamma_l\tau_{lt-2}) + \gamma_x\tau_{xt-2}) + \gamma_z \log z_{t-1} + \gamma_l\tau_{lt-1} + \gamma_x\tau_{xt-1} \\
&= \gamma_0[1 + \gamma_k + \gamma_k^2 + \ldots] \\
&\quad + \gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\quad + \gamma_l[\tau_{lt-1} + \gamma_k\tau_{lt-2} + \gamma_k^2\tau_{lt-3} + \ldots] \\
&\quad + \gamma_x[\tau_{xt-1} + \gamma_k\tau_{xt-2} + \gamma_k^2\tau_{xt-3} + \ldots]
\end{aligned}$$

or in differences as follows:

$$\begin{aligned}
\log \hat{k}_t - \log \hat{k}_{t-1} = \gamma_z&[\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
+ \gamma_l&[\tau_{lt-1} + (\gamma_k - 1)\{\tau_{lt-2} + \gamma_k\tau_{lt-3} + \gamma_k^2\tau_{lt-4} + \ldots\}] \\
+ \gamma_x&[\tau_{xt-1} + (\gamma_k - 1)\{\tau_{xt-2} + \gamma_k\tau_{xt-3} + \gamma_k^2\tau_{xt-4} + \ldots\}]
\end{aligned}$$

or in quasi-differences as follows:

$$\begin{aligned}\log \hat{k}_t - \alpha \log \hat{k}_{t-1} = \gamma_z[&\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\ &+ \gamma_l[\tau_{lt-1} + (\gamma_k - \alpha)\{\tau_{lt-2} + \gamma_k \tau_{lt-3} + \gamma_k^2 \tau_{lt-4} + \ldots\}] \\ &+ \gamma_x[\tau_{xt-1} + (\gamma_k - \alpha)\{\tau_{xt-2} + \gamma_k \tau_{xt-3} + \gamma_k^2 \tau_{xt-4} + \ldots\}]\end{aligned}$$

I can also write hours in terms of past shocks as follows:

$$\begin{aligned}\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_{lt} + \phi_{lx}\tau_{xt} + \phi_{lk} \log \hat{k}_t + \phi_{lk'} \log \hat{k}_{t+1} \\ &= \phi_{lz} \log z_t + \phi_{ll}\tau_{lt} + \phi_{lx}\tau_{xt} \\ &\quad + \phi_{lk}\gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\ &\qquad + \phi_{lk}\gamma_l[\tau_{lt-1} + \gamma_k \tau_{lt-2} + \gamma_k^2 \tau_{lt-3} + \ldots] \\ &\qquad + \phi_{lk}\gamma_x[\tau_{xt-1} + \gamma_k \tau_{xt-2} + \gamma_k^2 \tau_{xt-3} + \ldots] \\ &\quad + \phi_{lk'}\gamma_z[\log z_t + \gamma_k \log z_{t-1} + \gamma_k^2 \log z_{t-2} + \ldots] \\ &\qquad + \phi_{lk'}\gamma_l[\tau_{lt} + \gamma_k \tau_{lt-1} + \gamma_k^2 \tau_{lt-2} + \ldots] \\ &\qquad + \phi_{lk'}\gamma_x[\tau_{xt} + \gamma_k \tau_{xt-1} + \gamma_k^2 \tau_{xt-2} + \ldots] \\ &= [(\phi_{lz} + \phi_{lk'}\gamma_z) \log z_t + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_z \log z_{t-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_z \log z_{t-2} + \ldots] \\ &\quad + [(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_{lt} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_l\tau_{lt-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_l\tau_{lt-2} + \ldots] \\ &\quad + [(\phi_{ll} + \phi_{lk'}\gamma_x)\tau_{xt} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_x\tau_{xt-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_x\tau_{xt-2} + \ldots]\end{aligned}$$

where I have ignored constant terms.

I can write logged hours in differences as follows:

$$\begin{aligned}
\log l_t - \log l_{t-1} &= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_{lt} - \tau_{lt-1}) + \phi_{lx}(\tau_{xt} - \tau_{xt-1}) \\
&\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \log \hat{k}_{t-1}) \\
&= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_{lt} - \tau_{lt-1}) + \phi_{lx}(\tau_{xt} - \tau_{xt-1}) \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - 1)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_l[\tau_{lt} + (\gamma_k - 1)\{\tau_{lt-1} + \gamma_k \tau_{lt-2} + \gamma_k^2 \tau_{lt-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_x[\tau_{xt} + (\gamma_k - 1)\{\tau_{xt-1} + \gamma_k \tau_{xt-2} + \gamma_k^2 \tau_{xt-3} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_l[\tau_{lt-1} + (\gamma_k - 1)\{\tau_{lt-2} + \gamma_k \tau_{lt-3} + \gamma_k^2 \tau_{lt-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_x[\tau_{xt-1} + (\gamma_k - 1)\{\tau_{xt-2} + \gamma_k \tau_{xt-3} + \gamma_k^2 \tau_{xt-4} + \ldots\}] \\
&= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)] \log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_{lt} - [\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{lt-1} \\
&\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{lt-2} + \gamma_k \tau_{lt-3} + \gamma_k^2 \tau_{lt-4} + \ldots\} \\
&\quad + [\phi_{lx} + \phi_{lk'}\gamma_x]\tau_{xt} - [\phi_{lx} - \phi_{lk}\gamma_x - \phi_{lk'}\gamma_x(\gamma_k - 1)]\tau_{xt-1} \\
&\quad + \gamma_x(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{xt-2} + \gamma_k \tau_{xt-3} + \gamma_k^2 \tau_{xt-4} + \ldots\}
\end{aligned}$$

or in quasi-difference form as follows:

$$\begin{aligned}
\log l_t - \alpha \log l_{t-1} &= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_{lt} - \alpha \tau_{lt-1}) + \phi_{lx}(\tau_{xt} - \alpha \tau_{xt-1}) \\
&\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \alpha \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \alpha \log \hat{k}_{t-1}) \\
&= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_{lt} - \alpha \tau_{lt-1}) + \phi_{lx}(\tau_{xt} - \alpha \tau_{xt-1}) \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - \alpha)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_l[\tau_{lt} + (\gamma_k - \alpha)\{\tau_{lt-1} + \gamma_k \tau_{lt-2} + \gamma_k^2 \tau_{lt-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_x[\tau_{xt} + (\gamma_k - \alpha)\{\tau_{xt-1} + \gamma_k \tau_{xt-2} + \gamma_k^2 \tau_{xt-3} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_l[\tau_{lt-1} + (\gamma_k - \alpha)\{\tau_{lt-2} + \gamma_k \tau_{lt-3} + \gamma_k^2 \tau_{lt-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_x[\tau_{xt-1} + (\gamma_k - \alpha)\{\tau_{xt-2} + \gamma_k \tau_{xt-3} + \gamma_k^2 \tau_{xt-4} + \ldots\}] \\
&= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\alpha\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - \alpha)] \log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_{lt} - [\alpha\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - \alpha)]\tau_{lt-1} \\
&\quad + [\phi_{lx} + \phi_{lk'}\gamma_l]\tau_{xt} - [\alpha\phi_{lx} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_x(\gamma_k - \alpha)]\tau_{xt-1} \\
&\quad + \gamma_l(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{lt-2} + \gamma_k \tau_{lt-3} + \gamma_k^2 \tau_{lt-4} + \ldots\} \\
&\quad + \gamma_x(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{xt-2} + \gamma_k \tau_{xt-3} + \gamma_k^2 \tau_{xt-4} + \ldots\} \qquad (4.2)
\end{aligned}$$

I can use the expressions for output and hours to write out the change in productivity as follows:

$$
\begin{aligned}
\log(y_t/l_t) &- \log(y_{t-1}/l_{t-1}) \\
&= \log \hat{y}_t - \log \hat{y}_{t-1} + \log z_t - \log l_t - \log l_{t-1} \\
&= \log z_t + \theta(\log \hat{k}_t - \log \hat{k}_{t-1} - \log l_t + \log l_{t-1} - \log z_t + \log z_{t-1}) \\
&= (1-\theta)\log z_t + \theta \log z_{t-1} \\
&\quad - \theta(\log l_t - \log l_{t-1} - \log \hat{k}_t + \log \hat{k}_{t-1}) \\
&= (1-\theta)\log z_t + \theta \log z_{t-1} \\
&\quad - \theta(\log l_t - \log l_{t-1} - \log \hat{k}_t + \log \hat{k}_{t-1}) \\
&= (1-\theta)\log z_t + \theta \log z_{t-1} - \theta\{[\phi_{lz} + \phi_{lk'}\gamma_z]\log z_t \\
&\quad - [\phi_{lz} - (\phi_{lk}-1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)]\log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_{lt} - [\phi_{ll} - (\phi_{lk}-1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{lt-1} \\
&\quad + [\phi_{lx} + \phi_{lk'}\gamma_x]\tau_{xt} - [\phi_{lx} - (\phi_{lk}-1)\gamma_x - \phi_{lk'}\gamma_x(\gamma_k - 1)]\tau_{xt-1} \\
&\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{lt-2} + \gamma_k\tau_{lt-3} + \gamma_k^2 \log \tau_{lt-4} + \ldots]\} \\
&\quad + \gamma_x(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{xt-2} + \gamma_k\tau_{xt-3} + \gamma_k^2 \log \tau_{xt-4} + \ldots]\} \\
&= \{1 - \theta - \theta[\phi_{lz} + \phi_{lk'}\gamma_z]\}\log z_t \\
&\quad + \theta[1 + \phi_{lz} - (\phi_{lk}-1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)]\log z_{t-1} \\
&\quad - \theta\gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\quad - \theta[\phi_{ll} + \phi_{lk'}\gamma_l]\tau_{lt} \\
&\quad + \theta[\phi_{ll} - (\phi_{lk}-1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{lt-1} \\
&\quad - \theta\gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{lt-2} + \gamma_k\tau_{lt-3} + \gamma_k^2\tau_{lt-4} + \ldots] \\
&\quad - \theta[\phi_{lx} + \phi_{lk'}\gamma_x]\tau_{xt} \\
&\quad + \theta[\phi_{lx} - (\phi_{lk}-1)\gamma_x - \phi_{lk'}\gamma_x(\gamma_k - 1)]\tau_{xt-1} \\
&\quad - \theta\gamma_x(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{xt-2} + \gamma_k\tau_{xt-3} + \gamma_k^2\tau_{xt-4} + \ldots] \qquad (4.3)
\end{aligned}
$$

Now I write out the log of the investment share:

$$
\begin{aligned}
\log(x_t/y_t) &= \log \hat{x}_t - \log \hat{y}_t \\
&= \phi_{xk}(\log \hat{k}_t - \log z_t) + \phi_{xk'} \log \hat{k}_{t+1} - \theta(\log \hat{k}_t - \log z_t) - (1-\theta)\log l_t \\
&= (\phi_{xk} - \theta)(\log \hat{k}_t - \log z_t) + \phi_{xk'}[\gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_{lt} + \gamma_x \tau_{xt}] \\
&\quad - (1-\theta)[(\phi_{lk} + \phi_{lk'}\gamma_k)\log \hat{k}_t + (\phi_{lz} + \phi_{lk'}\gamma_z)\log z_t \\
&\qquad + (\phi_{ll} + \phi_{lk'}\gamma_l)\tau_{lt} + (\phi_{lx} + \phi_{lk'}\gamma_x)\tau_{xt}] \\
&= [-\phi_{xk} + \theta + \phi_{xk'}\gamma_z - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z)]\log z_t \\
&\quad + [\phi_{xk'}\gamma_l - (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)]\tau_{lt} \\
&\quad + [\phi_{xk'}\gamma_x - (1-\theta)(\phi_{lx} + \phi_{lk'}\gamma_x)]\tau_{xt} \\
&\quad + [\phi_{xk} - \theta + \phi_{xk'}\gamma_k - (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k)] \\
&\qquad \{\gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\qquad\quad + \gamma_l[\tau_{lt-1} + \gamma_k \tau_{lt-2} + \gamma_k^2 \tau_{lt-3} + \ldots] \\
&\qquad\quad + \gamma_x[\tau_{xt-1} + \gamma_k \tau_{xt-2} + \gamma_k^2 \tau_{xt-3} + \ldots]\}
\end{aligned}
$$

4.1.1. The Model's Moving Average

The moving average for the model is given by:

$$
\begin{bmatrix} (1-L)\log y_t/l_t \\ (1-\alpha L)\log l_t \\ \log x_t/y_t \end{bmatrix} \equiv X_t = D_0\omega_t + D_1\omega_{t-1} + D_2\omega_{t-2} + \ldots
$$

where $\omega_t = [\log z_t, \tau_{lt}, \tau_{xt}]'$ and

$$
D_0 = \begin{bmatrix} 1-\theta+\theta a & -\theta b & -\theta c \\ -a & b & c \\ -d & e & f \end{bmatrix} \tag{4.4}
$$

$$
D_1 = \begin{bmatrix} \theta(1-a)(1-\gamma_k) & \theta(b+(1-a)\gamma_l) & \theta(c+(1-a)\gamma_x) \\ (\alpha-\gamma_k)a & -\alpha b + \gamma_l a & -\alpha c + \gamma_x a \\ -d\gamma_k & d\gamma_l & d\gamma_x \end{bmatrix} \tag{4.5}
$$

$$
D_2 = \begin{bmatrix} \gamma_k(1-a)\theta(1-\gamma_k) & -\gamma_l(1-a)\theta(1-\gamma_k) & -\gamma_x(1-a)\theta(1-\gamma_k) \\ \gamma_k a(\alpha-\gamma_k) & -\gamma_l a(\alpha-\gamma_k) & -\gamma_x a(\alpha-\gamma_k) \\ -d\gamma_k^2 & d\gamma_l\gamma_k & d\gamma_x\gamma_k \end{bmatrix}, \tag{4.6}
$$

and $D_j = \gamma_k D_{j-1}$ for $j \geq 3$ where $a = \phi_{lk} + \phi_{lk'}\gamma_k$, $b = \phi_{ll} + \phi_{lk'}\gamma_l$, $c = \phi_{lx} + \phi_{lk'}\gamma_x$, $d = \phi_{xk} + \phi_{xk'}\gamma_k - \theta - (1-\theta)a$, $e = \phi_{xk'}\gamma_l - (1-\theta)b$, and $f = \phi_{xk'}\gamma_x - (1-\theta)c$. Note that $\phi_{lz} = -\phi_{lk}$, $\phi_{xz} = -\phi_{xk}$, and $\gamma_z = -\gamma_k$ hold in the model economy with a unit root in technology. Note also that D_2 is singular for all parameterizations, and D_1 is singular if $\alpha = 0$.

If τ_{lt} and τ_{xt} are AR(1) processes, then it is more convenient to write the MA process in terms of $\eta_t = [\log z_t, \eta_{lt}, \eta_{xt}]$ rather than in terms of ω_t. In this case,

$$X_t = D_0\eta_t + (D_0P + D_1)\eta_{t-1} + (D_0P^2 + D_1P + D_2)\eta_{t-2} + (D_0P^3 + D_1P^2 + D_2P + D_3)\eta_{t-3}\cdots.$$

I normalize the MA so it has an identity for the first coefficient. That is, I set $C_0 = I$, $C_1 = (D_0P + D_1)D_0^{-1}$, and $C_j = C_{j-1}D_0PD_0^{-1} + D_jD_0^{-1}$.

4.1.2. Special Property of the D's

As in the 2-shock case, it is the case that the D matrices have a special property that can be exploited when I characterize coefficients of the VAR of X_t. In other words, the D's for the 3-shock RBC model also satisfy the relation:

$$\left(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1}\right) D_2 = 0. \tag{4.7}$$

which is the same as (3.1). Because D_1 is singular when $\alpha = 0$, I will assume that the choice of P and α is such that $D_0P + D_1$ is invertible. This rules out the case with P and α identically equal to 0. If that is the case of interest, assume that α is positive but very close to zero.

The steps of the proof of (4.7) in the 3-variable case is the same as in the 2-variable case. First note from (4.6) that

$$D_2 = \begin{bmatrix} (1-a)\theta(1-\gamma_k) \\ (\alpha - \gamma_k)a \\ -d\gamma_k \end{bmatrix} [\,\gamma_k \quad -\gamma_l \quad -\gamma_x\,] \equiv gh'$$

Thus, I can rewrite the left hand side as follows

$$\begin{aligned}&\left(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1}\right)D_2 \\ &\quad = \left[\gamma_k(gh') - (gh')(D_0P + D_1)^{-1}(gh')\right] - \left[(D_0P + D_1)P(D_0P + D_1)^{-1}gh'\right].\end{aligned} \tag{4.8}$$

Both terms in (4.8) in square brackets are equal to 3×3 zero matrices. The first step is to show that

$$(D_0P + D_1)^{-1}g = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.9}$$

The proof of this step is trivial since the first column of D_0P+D_1 is equal to g. Substituting (4.9) into (4.8), the result (4.7) follows immediately from the fact that $h'[1,0,0]' = \gamma_k$ and $P[1,0,0]' = 0$.

4.1.3. Proposition 4: Model has infinite-order VAR

The map between the theoretical MA and the VAR is the same as before. What is new is the VAR representation.

Proposition 4. The model described above has a VAR representation with coefficients B_j that satisfy

$$B_j = MB_{j-1} \tag{4.10}$$

for $j \geq 2$, with $B_1 = C_1 = (D_0P + D_1)D_0^{-1}$. The matrix M is 3×3 with eigenvalues equal to 0, α, and $(1-\delta)/[z(1+g_n)]$.

Proof of Proposition 4. The first part of the proof is the same as for Proposition 1. The second part, involving the expressions of the eigenvalues, is different. In the three shock case, one can use the same derivations as those in Proposition 1 to show that $[1,0,0]'h' - D_0^{-1}D_1$ has the same eigenvalues as M. In this case, D_0^{-1} is given by:

$$D_0^{-1} = \frac{1}{|D_0|}\begin{bmatrix} bf-ce & \theta(bf-ce) & 0 \\ af-cd & (1-\theta)f+\theta(af-cd) & -(1-\theta)c \\ bd-ae & -(1-\theta)e+\theta(bd-ae) & (1-\theta)b \end{bmatrix}$$

and the elements of $[1,0,0]'h' - D_0^{-1}D_1$ are given by

$$\begin{aligned}
(1,1) &= \gamma_k - \theta(1-\gamma_k - a + a\alpha)/(1-\theta) \\
(1,2) &= -\gamma_l - \theta(\gamma_l + b - b\alpha)/(1-\theta) \\
(1,3) &= -\gamma_x - \theta(\gamma_x + c - c\alpha)/(1-\theta) \\
(2,1) &= \{(af-cd)(\gamma_k - \theta + \theta a - \theta a\alpha) - af(1-\theta)\alpha]\}/|D_0| \\
(2,2) &= \{(af-cd)(-\gamma_l - \theta b + \theta b\alpha) + bf(1-\theta)\alpha\}/|D_0| \\
(2,3) &= \{(af-cd)(-\gamma_x - \theta c + \theta c\alpha) + cf(1-\theta)\alpha\}/|D_0| \\
(3,1) &= \{(bd-ae)(\gamma_k - \theta + \theta a - \theta a\alpha) + ae(1-\theta)\alpha\}/|D_0| \\
(3,2) &= \{(bd-ae)(-\gamma_l - \theta b + \theta b\alpha) - be(1-\theta)\alpha\}/|D_0| \\
(3,3) &= \{(bd-ae)(-\gamma_x - \theta c + \theta c\alpha) - ce(1-\theta)\alpha\}/|D_0|
\end{aligned}$$

where $|D_0| = (1-\theta)(bf-ce)$. To prove the proposition, I will show that $trace([1,0,0]'h' - D_0^{-1}D_1)$ equals the sum of the proposed eigenvalues, $|[1,0,0]'h' - D_0^{-1}D_1| = 0$, and $|[1,0,0]'h' - D_0^{-1}D_1 - \alpha I| = 0$. These three conditions uniquely determine the three eigenvalues.

To compute the trace, sum (1,1), (2,2), and (3,3):

$$\text{trace}([1,0,0]'h' - D_0^{-1}D_1) = \gamma_k - \theta(1-\gamma_k - a + a\alpha)/(1-\theta)$$

$$
\begin{aligned}
&+ \{(af-cd)(-\gamma_l-\theta b+\theta b\alpha)+bf(1-\theta)\alpha \\
&\quad + (bd-ae)(-\gamma_x-\theta c+\theta c\alpha)-ce(1-\theta)\alpha\}/|D_0| \\
&= \alpha + \{(\gamma_k-\theta)(bf-ce)-\gamma_l(af-cd)-\gamma_x(bd-ae)\}/|D_0| \\
&= \alpha + \{(\gamma_k-\theta)\phi_{xk'}(b\gamma_x-c\gamma_l)-\gamma_l(a\phi_{xk'}\gamma_x-c(\phi_{xk}+\phi_{xk'}\gamma_k-\theta)) \\
&\quad - \gamma_x(b(\phi_{xk}+\phi_{xk'}\gamma_k-\theta)-a\phi_{xk'}\gamma_l)\} \\
&\qquad /[(1-\theta)\phi_{xk'}(b\gamma_x-c\gamma_l)]
\end{aligned}
$$

$$
= \alpha + \frac{\theta(1-\phi_{xk'})-\phi_{xk}}{\phi_{xk'}(1-\theta)} \tag{4.11}
$$

$$
= \alpha + \frac{(1-\delta)}{z(1+g_n)} \tag{4.12}
$$

where z without a subscript is the steady state value.

Next I compute the determinant of $[1,0,0]'h' - D_0^{-1}D_1$ and show it is 0. Denoting the matrix by $\mathcal{M}$, I get

$$
\begin{aligned}
\det(\mathcal{M}) &= \mathcal{M}_{1,1}|\mathcal{M}([2,3],[2,3])| - \mathcal{M}_{1,2}|\mathcal{M}([2,3],[1,3])| + \mathcal{M}_{1,3}|\mathcal{M}([2,3],[1,2])| \\
&= (\gamma_k - \theta(1-\gamma_k-a+a\alpha)/(1-\theta))(1-\theta)\alpha d(fb-ec)(\gamma_l c-\gamma_x b)/|D_0|^2 \\
&\quad + (\gamma_l + \theta(\gamma_l+b-b\alpha)/(1-\theta))(1-\theta)\alpha d(fb-ec)(-\gamma_k c+\theta c+\gamma_x a)/|D_0|^2 \\
&\quad - (\gamma_x + \theta(\gamma_x+c-c\alpha)/(1-\theta))(1-\theta)\alpha d(fb-ec)(-\gamma_k b+\theta b+\gamma_l a)/|D_0|^2 \\
&= \big\{(\gamma_k(1-\theta)-\theta(1-\gamma_k-a+a\alpha))(\gamma_l c-\gamma_x b) \\
&\quad + (\gamma_l(1-\theta)+\theta(\gamma_l+b-b\alpha))(-\gamma_k c+\theta c+\gamma_x a) \\
&\quad - (\gamma_x(1-\theta)+\theta(\gamma_x+c-c\alpha))(-\gamma_k b+\theta b+\gamma_l a)\big\}\alpha d(fb-ec)/|D_0|^2 \\
&= \big\{(\gamma_k-\theta+\theta a(1-\alpha))(\gamma_l c-\gamma_x b) \\
&\quad + (\gamma_l+\theta b(1-\alpha))(-(\gamma_k-\theta)c+\gamma_x a) \\
&\quad - (\gamma_x+\theta c(1-\alpha))(-(\gamma_k-\theta)b+\gamma_l a)\big\}\alpha d(fb-ec)/|D_0|^2 \\
&= 0
\end{aligned} \tag{4.13}
$$

Finally, I take the determinant of $\mathcal{M} - \alpha I$ and show it is 0 as follows:

$$
\begin{aligned}
&\det(\mathcal{M}-\alpha I) \\
&\quad = (\mathcal{M}_{1,1}-\alpha)|\mathcal{M}([2,3],[2,3])-\alpha I| - \mathcal{M}_{1,2}(|\mathcal{M}([2,3],[1,3])| - \alpha\mathcal{M}_{2,1}) \\
&\qquad + \mathcal{M}_{1,3}(|\mathcal{M}([2,3],[1,2])| + \alpha\mathcal{M}_{3,1}) \\
&\quad = \alpha\{\alpha[\mathcal{M}_{1,1}+\mathcal{M}_{2,2}+\mathcal{M}_{3,3}-\alpha]
\end{aligned}
$$

$$
\begin{aligned}
&- \mathcal{M}_{1,1}\mathcal{M}_{2,2} - \mathcal{M}_{1,1}\mathcal{M}_{3,3} - \mathcal{M}_{2,2}\mathcal{M}_{3,3} \\
&+ \mathcal{M}_{1,2}\mathcal{M}_{2,1} + \mathcal{M}_{1,3}\mathcal{M}_{3,1} + \mathcal{M}_{2,3}\mathcal{M}_{3,2}\} \\
= {} & \alpha\{\alpha[\text{trace}(\mathcal{M}) - \alpha] \\
&- (\mathcal{M}_{1,1}\mathcal{M}_{2,2} - \mathcal{M}_{1,2}\mathcal{M}_{2,1}) \\
&- (\mathcal{M}_{1,1}\mathcal{M}_{3,3} - \mathcal{M}_{1,3}\mathcal{M}_{3,1}) \\
&- (\mathcal{M}_{2,2}\mathcal{M}_{3,3} - \mathcal{M}_{2,3}\mathcal{M}_{3,2})\} \\
= {} & \alpha\{\alpha[(bf - ce)(\gamma_k - \theta + \theta a - \theta a\alpha) \\
&\quad + (af - cd)(-\gamma_l - \theta b + \theta b\alpha) + bf(1-\theta)\alpha \\
&\quad + (bd - ae)(-\gamma_x - \theta c + \theta c\alpha) - ce(1-\theta)\alpha - \alpha(1-\theta)(bf - ce)] \\
&- f\alpha[b(\gamma_k - \theta) - a\gamma_l] \\
&+ e\alpha[c(\gamma_k - \theta) - a\gamma_x] \\
&- d\alpha[\gamma_l c - \gamma_x b]\}/|D_0| \\
= {} & 0 \qquad (4.14)
\end{aligned}
$$

The result in (4.14) implies that α is an eigenvalue. The result in (4.13) implies that 0 is an eigenvalue. Given these results, the fact that the trace is (4.12) implies that $(1-\delta)/[z(1+g_n)]$ is the third eigenvalue. This completes the proof. ■

4.1.4. A Way to Make M Singular

Above I included the investment share in the VAR. The investment share is typically added to capture the capital dynamics if capital is unobserved. What if I assume that capital is observed and use the capital share instead?

I can see the answer directly from the proof in Proposition 4. At the step (equation (4.11)) that I fill in expressions for ϕ_{xk} and $\phi_{xk'}$ using (2.18), I could instead use $\phi_{xk} = 1$ and $\phi_{xk'} = 0$. This yields a third eigenvalue equal to -1/0 or $-\infty$. This clearly doesn't work since the MA is not invertible.

However, it shows me what would work: adding the capital *next period* relative to output, $\log(k_{t+1}/y_t)$, and, therefore, setting $\phi_{xk} = 0$ and $\phi_{xk'} = 1$. If $\alpha = 0$, then the matrix M has 3 zero eigenvalues. A researcher running a VAR would find that B_2 is singular and the rest of the B_j, $j \geq 3$, are zero matrices. In fact the structure would be such that the second and third column of B_2 would be equal and equal to the negative of the first column of B_2. That is how it works: certain lags are cancelling so it effectively mimics the model's finite-lag VAR.

What is interesting is that it won't work if I divide k_{t+1} by k_t and include the log of the growth rate of capital. If I add $\log(k_{t+1}/k_t)$ to the VAR, then I proceed the same way through the proof of Proposition 4 using $d = \gamma_k - 1$, $e = \gamma_l$, and $f = \gamma_x$. The result is that the eigenvalues of M are 0, α, and 1. The fact that one is 1 means that the MA is not invertible.

What these results tell me is that one has to proceed carefully, using lots of the details of the model, to determine if the SVAR has a short-lag representation. Since most business cycle models have a short-lag state-space representation, we advise using it directly. The state-space representation also allows us to treat the capital stocks as unobserved. This is certainly necessary in business cycle models with sticky price models and staggered constracts; the state vector includes the distribution of capital stocks which is unobserved.

4.2. Government Spending Shock and Investment-Output in VAR

Assume the economy is an RBC model with three orthogonal shocks: a unit root in technology $\log z$, an AR(1) in the tax rate on labor τ_l, and an AR(1) in $\log \hat{g}_t$. The capital decision function has the form:

$$\log \hat{k}_{t+1} = \gamma_0 + \gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_{lt} + \gamma_g \log \hat{g}_t.$$

4.2.1. The Model's Moving Average

I can borrow the expressions from above for this case, replacing γ_x by γ_g and τ_{xt} by $\log \hat{g}_t$. In other words, the D matrices are given by:

$$D_0 = \begin{bmatrix} 1-\theta+\theta a & -\theta b & -\theta c \\ -a & b & c \\ -d & e & f \end{bmatrix} \tag{4.15}$$

$$D_1 = \begin{bmatrix} \theta(1-a)(1-\gamma_k) & \theta(b+(1-a)\gamma_l) & \theta(c+(1-a)\gamma_g) \\ (\alpha-\gamma_k)a & -\alpha b+\gamma_l a & -\alpha c+\gamma_g a \\ -d\gamma_k & d\gamma_l & d\gamma_g \end{bmatrix} \tag{4.16}$$

$$D_2 = \begin{bmatrix} \gamma_k(1-a)\theta(1-\gamma_k) & -\gamma_l(1-a)\theta(1-\gamma_k) & -\gamma_g(1-a)\theta(1-\gamma_k) \\ \gamma_k a(\alpha-\gamma_k) & -\gamma_l a(\alpha-\gamma_k) & -\gamma_g a(\alpha-\gamma_k) \\ -d\gamma_k^2 & d\gamma_l\gamma_k & d\gamma_g\gamma_k \end{bmatrix}, \tag{4.17}$$

and $D_j = \gamma_k D_{j-1}$ for $j \geq 3$ where $a = \phi_{lk} + \phi_{lk'}\gamma_k$, $b = \phi_{ll} + \phi_{lk'}\gamma_l$, $c = \phi_{lg} + \phi_{lk'}\gamma_g$, $d = \phi_{xk} + \phi_{xk'}\gamma_k - \theta - (1-\theta)a$, $e = \phi_{xk'}\gamma_l - (1-\theta)b$, and $f = \phi_{xk'}\gamma_g - (1-\theta)c$. Again, it is the case that $\phi_{lz} = -\phi_{lk}$, $\phi_{xz} = -\phi_{xk}$, and $\gamma_z = -\gamma_k$. Note also that D_2 is singular for all parameterizations, and D_1 is singular if $\alpha = 0$.

As before, we can rewrite the theoretical MA in terms of $\eta_t = [\log z_t, \eta_{lt}, \eta_{gt}]$ where this time, the third shock is government spending shock η_{gt}.

Because the D matrices have the same form as in the case with the investment tax shock, it is trivial to prove that Proposition 4 also holds for the economy with government spending shocks. The matrix M in the two cases are different (i.e., have different elements) but the eigenvalues are the same. The C matrices (where $C_j = C_{j-1} D_0 P D_0^{-1} + D_j D_0^{-1}$) also have different elements. So, the garbled terms in $\bar{C} = I + C_1 + C_2 + \ldots$ and the variance-covariance matrix Evv' will differ depending on which shock I use. What Proposition 4 tells me is that the garbling will likely imply that the mistaken inference is quantitatively important if the third shock has a nonnegligible variance.

4.3. Investment Tax Shock and Consumption-Output in VAR

The decision function for next period capital is again (4.1), the formula for the change in productivity is again (4.3), and the formula for quasi-differenced hours is (4.2).

Suppose the third variable in the VAR is the log of the consumption share:

$$\begin{aligned}
\log(c_t/y_t) &= \log \hat{c}_t - \log \hat{y}_t \\
&= (\phi_{ck} - (1-\theta)\phi_{lk} - \theta) \log \hat{k}_t + (\phi_{cz} - (1-\theta)\phi_{lz} + \theta) \log z_t \\
&\quad + (\phi_{cl} - (1-\theta)\phi_{ll})\tau_{lt} + (\phi_{cx} - (1-\theta)\phi_{lx})\tau_{xt} \\
&\quad + (\phi_{ck'} - (1-\theta)\phi_{lk'}) \log \hat{k}_{t+1} \\
&= [\phi_{ck} + \phi_{ck'}\gamma_k - (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k) - \theta] \log \hat{k}_t \\
&\quad + [\phi_{cz} + \phi_{ck'}\gamma_z - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z) + \theta] \log z_t \\
&\quad + [\phi_{cl} + \phi_{ck'}\gamma_l - (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)]\tau_{lt} \\
&\quad + [\phi_{cx} + \phi_{ck'}\gamma_x - (1-\theta)(\phi_{lx} + \phi_{lk'}\gamma_x)]\tau_{xt} \\
&= [\phi_{cz} + \phi_{ck'}\gamma_z - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z) + \theta] \log z_t \\
&\quad + [\phi_{cl} + \phi_{ck'}\gamma_l - (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)]\tau_{lt} \\
&\quad + [\phi_{cx} + \phi_{ck'}\gamma_l - (1-\theta)(\phi_{lx} + \phi_{lk'}\gamma_x)]\tau_{xt} \\
&\quad + [\phi_{ck} + \phi_{ck'}\gamma_k - (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k) - \theta] \\
&\qquad \{\gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\qquad\quad + \gamma_l[\tau_{lt-1} + \gamma_k \tau_{lt-2} + \gamma_k^2 \tau_{lt-3} + \ldots] \\
&\qquad\quad + \gamma_x[\tau_{xt-1} + \gamma_k \tau_{xt-2} + \gamma_k^2 \tau_{xt-3} + \ldots]\}
\end{aligned}$$

4.3.1. The Model's Moving Average

The moving average can again be represented in terms of the D matrices in (4.4)-(4.6) with $D_j = \gamma_k D_{j-1}$ for $j \geq 3$ and $a = \phi_{lk} + \phi_{lk'}\gamma_k$, $b = \phi_{ll} + \phi_{lk'}\gamma_l$, $c = \phi_{lx} + \phi_{lk'}\gamma_x$, $d = \phi_{ck} + \phi_{ck'}\gamma_k - \theta - (1-\theta)a$, $e = \phi_{cl} + \phi_{ck'}\gamma_l - (1-\theta)b$, and $f = \phi_{cx} + \phi_{ck'}\gamma_x - (1-\theta)c$. Note that $\phi_{lz} = -\phi_{lk}$, $\phi_{cz} = -\phi_{ck}$, and $\gamma_z = -\gamma_k$ hold in the model economy with a unit root in technology. Note also that D_2 is singular for all parameterizations, and D_1 is singular if $\alpha = 0$.

Because the D matrices have the same form, I can again find the eigenvalues of M by working with the simpler matrix $[1,0,0]'h' - D_0^{-1}D_1$. I will show that the eigenvalues in this case are 0, α, and

$$\frac{1-\delta}{z(1+g_n)} + \left(\frac{\theta}{1-\theta}\right)\frac{\hat{g}}{\hat{k}(1+g_n)}$$

I can follow the proof of proposition 8 to show this. There is only one deviation needed, which is the derivation of the third eigenvalue. Writing out the trace of $[1,0,0]'h' - D_0^{-1}D_1$ yields

$$\text{trace}([1,0,0]'h' - D_0^{-1}D_1) = \alpha + \{(\gamma_k - \theta)(bf - ce) - \gamma_l(af - cd) - \gamma_x(bd - ae)\}/|D_0|$$

as before. Using the new arguments of the D's, this trace can be simplified as follows:

$$\begin{aligned}
&\text{trace}([1,0,0]'h' - D_0^{-1}D_1)\\
&\quad = \alpha - \frac{\theta}{1-\theta} + \{\gamma_k(b(\phi_{cx} + \phi_{ck'}\gamma_x) - c(\phi_{cl} + \phi_{ck'}\gamma_l))\\
&\qquad\quad - \gamma_l(a(\phi_{cx} + \phi_{ck'}\gamma_x) - c(\phi_{ck} + \phi_{ck'}\gamma_k - \theta))\\
&\qquad\quad - \gamma_x(b(\phi_{ck} + \phi_{ck'}\gamma_k - \theta) - a(\phi_{cl} + \phi_{ck'}\gamma_l))\}/|D_0|\\
&\quad = \alpha - \frac{\theta}{1-\theta} + \{-\gamma_k c\phi_{cl} + \gamma_l c(\phi_{ck} - \theta)\\
&\qquad\quad - \gamma_x(b(\phi_{ck} - \theta) - a\phi_{cl})\}/|D_0|\\
&\quad = \alpha - \frac{\theta}{1-\theta} + \{-\gamma_k\phi_{lk'}\gamma_x\phi_{cl} + \gamma_l\phi_{lk'}\gamma_x(\phi_{ck} - \theta)\\
&\qquad\quad - \gamma_x(\phi_{ll} + \phi_{lk'}\gamma_l)(\phi_{ck} - \theta) + \gamma_x(\phi_{lk} + \phi_{lk'}\gamma_k)\phi_{cl})\}/|D_0|\\
&\quad = \alpha - \frac{\theta}{1-\theta} + \gamma_x\{\phi_{ll}\theta - \phi_{ll}\phi_{ck} + \phi_{lk}\phi_{cl}\}/|D_0|\\
&\quad = \alpha - \frac{\theta}{1-\theta} + \frac{\phi_{ll}\theta - \phi_{ll}\phi_{ck} + \phi_{lk}\phi_{cl}}{(1-\theta)(\phi_{ll}\phi_{ck'} - \phi_{cl}\phi_{lk'}}\\
&\quad = \alpha + \frac{1-\delta}{z(1+g_n)} + \left(\frac{\theta}{1-\theta}\right)\frac{\hat{g}}{\hat{k}(1+g_n)}
\end{aligned}$$

where I am using $\phi_{cl} = (1-\theta)\phi_{ll}\hat{y}/\hat{c}$, $\phi_{ck'} = (1-\theta)\phi_{lk'}\hat{y}/\hat{c} - (1+g_n)\hat{k}/\hat{c}$, and $\phi_{ck} = (1-\theta)\phi_{lk}\hat{y}/\hat{c} + \theta\hat{y}/\hat{c} + (1-\delta)\hat{k}/(z\hat{c})$ to get the final result. The mathematics for the other steps of the proof are the same as in the case where investment-output is the third variable in the VAR.

I also considered a government spending shock with the third variable being the consumption-output ratio. The maximal eigenvalue of M is not the same as in the case of the investment tax shock because ϕ_{lg} and ϕ_{cg} is not zero. In fact the expression for the eigenvalue is messy and uninterpretable. I evaluated it numerically and found it large for reasonable parameters. This means that the garbling will be significant no matter what shock is included.

4.4. Adding Variables with Little Effect on Fluctuations

Conventional wisdom says that researchers interested in the effects of a particular shock, say technology, can ignore specifying anything about the other shocks. In this section, I use an example from above—the 3-variable example with investment-to-output in the VAR and a shock to the investment tax—to illustrate why one cannot ignore the other shocks *even if they have a small effect on the variances of the observables.*

Consider two economies. The first has a capital decision function with γ_x as the coefficient on τ_{xt} and a variance for the innovation equal to η_x. The second has $\tilde{\gamma}_x$ and $\tilde{\eta}_x$ with $\tilde{\eta}_x$ chosen so that the following holds: $\gamma_x\eta_x = \tilde{\gamma}_x\tilde{\eta}_x$. Compute variance decompositions for the system

$$S_{t+1} = A_S S_t + B_S \eta_t$$
$$X_t = C_S S_t$$

with state vector $S_t = [\log \hat{k}_t, \log z_t, \tau_{lt}, \tau_{xt}, \log \hat{k}_{t-1}, \log z_{t-1}, \tau_{lt-1}, \tau_{xt-1}, 1]'$ and innovations $\eta_t = [0, \eta_{zt}, \eta_{lt}, \eta_{xt}, 0, 0, 0, 0, 0]'$ for the two economies. Also compute impulse responses applying the Blanchard-Quah methodology. Call these impulse responses the "empirical" impulse responses.

Proposition 5. The variance decompositions and empirical impulse responses are the same for the economy with γ_x and η_x as those for the economy with $\tilde{\gamma}_x$ and $\tilde{\eta}_x$.

Proof of Proposition 5. The two economies have the same variances and covariances for the observed variables. That implies that the variance decomposition calculations are the same in the two economies. It also implies that the variance covariance of errors in a VAR of X on lags is the same. Thus, to prove the proposition, I need only show that the C matrices (and hence M) are not affected by the size of γ_x. To do this, I can write out the elements of $D_0PD_0^{-1}$, $D_1D_0^{-1}$, and $D_2D_0^{-1}$ and show that they do not depend on γ_x. This is simple algebra and left for the reader. ■

Proposition 5 contradicts the view that "other shocks don't matter." In the example above, it could be that τ_{xt} contributes little to the total variation in X_t. But, if the eigenvalues in M are near 1, the garbled term in the expression for $\bar{C}$ (see (3.25)) will still be large. This is what happens with shocks like τ_{xt} and $\log \hat{g}_t$. They are not very important in accounting for business cycle fluctuations, but their variances are nonnegligible. Thus, they garble the signal of the other shocks.

There is also another point that should be made here. The other shocks may not be the "deep structural" shocks that researchers desire. In other words, likelihood ratio tests would reject the hypothesis that our shocks are uncorrelated. This is (maybe) a more important reason for not ignoring the other shocks.

5. VARs and 4-Shock Versions of the Model

In this section I consider a VAR with 4 variables and 4 shocks. The four variables are the change in the log of labor productivity, quasi-differenced logged hours, the log of the investment-output ratio, and the log of the consumption-output ratio.

Proposition 6. The model just described has a VAR representation with coefficients B_j that satisfy

$$B_j = MB_{j-1}$$

for $j \geq 2$, with $B_1 = C_1 = (D_0P + D_1)D_0^{-1}$. The matrix M is 4×4 with eigenvalues equal to 0, 0, α, and $(1-\delta)/[z(1+g_n)]$.

Proof of Proposition 6. The first part of the proof, showing that the VAR has an infinite number of lags, follows exactly as in Proposition 1. The second part, which involves derivation of eigenvalues of M, is almost exactly as in Proposition 4. In this case, however, I need to work with different D matrices. The matrices are given by

$$D_0 = \begin{bmatrix} 1-\theta+\theta a & -\theta b & -\theta c & -\theta d \\ -a & b & c & d \\ -e & f & g & h \\ -i & j & k & l \end{bmatrix} \tag{5.1}$$

$$D_1 = \begin{bmatrix} \theta(1-a)(1-\gamma_k) & \theta(b+(1-a)\gamma_l) & \theta(c+(1-a)\gamma_x) & \theta(d+(1-a)\gamma_g) \\ (\alpha-\gamma_k)a & -\alpha b+\gamma_l a & -\alpha c+\gamma_x a & -\alpha d+\gamma_g a \\ -e\gamma_k & e\gamma_l & e\gamma_x & e\gamma_g \\ -i\gamma_k & i\gamma_l & i\gamma_x & i\gamma_g \end{bmatrix} \tag{5.2}$$

$$D_2 = \begin{bmatrix} \theta(1-a)(1-\gamma_k) \\ (\alpha-\gamma_k)a \\ -e\gamma_k \\ -i\gamma_k \end{bmatrix} [\,\gamma_k \quad -\gamma_l \quad -\gamma_x \quad -\gamma_g\,] \equiv mn' \tag{5.3}$$

and $D_j = \gamma_k D_{j-1}$ for $j \geq 3$ where

$$\begin{aligned}
a &= \phi_{lk} + \phi_{lk'}\gamma_k \\
b &= \phi_{ll} + \phi_{lk'}\gamma_l \\
c &= \phi_{lk'}\gamma_x \\
d &= \phi_{lg} + \phi_{lk'}\gamma_g \\
e &= \phi_{xk} + \phi_{xk'}\gamma_k - \theta - (1-\theta)a \\
f &= \phi_{xk'}\gamma_l - (1-\theta)b \\
g &= \phi_{xk'}\gamma_x - (1-\theta)c \\
h &= \phi_{xk'}\gamma_g - (1-\theta)d \\
i &= \phi_{ck} + \phi_{ck'}\gamma_k - \theta - (1-\theta)a \\
j &= \phi_{cl} + \phi_{ck'}\gamma_l - (1-\theta)b \\
k &= \phi_{ck'}\gamma_x - (1-\theta)c \\
l &= \phi_{cg} + \phi_{ck'}\gamma_g - (1-\theta)d
\end{aligned}$$

Note that $\phi_{lz} = -\phi_{lk}$, $\phi_{xz} = -\phi_{xk}$, $\phi_{cz} = -\phi_{ck}$, and $\gamma_z = -\gamma_k$ hold in the model economy with a unit root in technology. Note also that D_2 is singular for all parameterizations, and D_1 is singular if $\alpha = 0$.

Using the same proof as in Proposition 1 or Proposition 4, it is easy to show that the eigenvalues of M are equal to the eigenvalues of

$$\mathcal{M} = [1,0,0,0]'n' - D_0^{-1}D_1.$$

Therefore, to prove Proposition 6, I need to prove that $|\mathcal{M} - \alpha I| = 0$, $|\mathcal{M}| = 0$, $|\mathcal{M} - (1-\delta)/(z(1+g_n))I| = 0$, and $\text{trace}(\mathcal{M}) = \alpha + (1-\delta)/(z(1+g_n))$.

Let me start with the inverse D_0^{-1}:

$$\frac{1}{|D_0|}\begin{bmatrix} |D_0|/(1-\theta) & \theta|D_0|/(1-\theta) & 0 & 0 \\ 0 & (1-\theta)(gl-hk) & 0 & (1-\theta)(ch-dg) \\ \begin{matrix} a(hj-fl)+b(el-hi) \\ +d(fi-ej) \end{matrix} & \begin{matrix} (1-\theta)(hj-fl) \\ +\theta\{a(hj-fl)+b(el-hi)+d(fi-ej)\} \end{matrix} & (1-\theta)(bl-dj) & (1-\theta)(df-bh) \\ \begin{matrix} a(fk-gj)+b(gi-ek) \\ +c(ej-fi) \end{matrix} & \begin{matrix} (1-\theta)(fk-gj) \\ +\theta\{a(fk-gj)+b(gi-ek)+c(ej-fi)\} \end{matrix} & (1-\theta)(cj-bk) & (1-\theta)(bg-cf) \end{bmatrix}$$

where $|D_0| = (1-\theta)(bgl - bhk + chj - dgj)$. In calculating this inverse, I use the following facts:

$$
\begin{aligned}
0 &= a(gl - hk) + c(hi - el) + d(ek - gi) \\
0 &= a(ch - dg) + c(ed - ah) + d(ag - ce) \\
0 &= a(bl - dj) + b(di - al) + d(aj - bi) \\
0 &= a(bh - df) + b(de - ah) + d(af - be) \\
0 &= cl - dk.
\end{aligned}
$$

Using D_0^{-1}, I have then that the elements of $\mathcal{M} = [1,0,0,0]'n' - D_0^{-1}D_1$ are

$$
\begin{aligned}
\mathcal{M}_{1,1} &= \gamma_k - \theta(1 - \gamma_k - a + \alpha a)/(1-\theta) \\
\mathcal{M}_{1,2} &= -\gamma_l - \theta(\gamma_l + b - b\alpha)/(1-\theta) \\
\mathcal{M}_{1,3} &= -\gamma_x - \theta(\gamma_x + c - c\alpha)/(1-\theta) \\
\mathcal{M}_{1,4} &= -\gamma_g - \theta(\gamma_g + d - d\alpha)/(1-\theta) \\
\mathcal{M}_{2,1} &= (1-\theta)\{(hk - gl)(a\alpha - \gamma_k a) + (ch - dg)i\gamma_k\}/|D_0| \\
\mathcal{M}_{2,2} &= (1-\theta)\{(hk - gl)(-\alpha b + \gamma_l a) - (ch - dg)i\gamma_l\}/|D_0| \\
\mathcal{M}_{2,3} &= (1-\theta)\{(hk - gl)(-\alpha c + \gamma_x a) - (ch - dg)i\gamma_x\}/|D_0| \\
\mathcal{M}_{2,4} &= (1-\theta)\{(hk - gl)(-\alpha d + \gamma_g a) - (ch - dg)i\gamma_g\}/|D_0| \\
\mathcal{M}_{3,1} &= \{[a(hj - fl) + b(el - hi) + d(fi - ej)][\gamma_k - \theta(1 - a + a\alpha)] - a(hj - fl)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{3,2} &= \{[a(hj - fl) + b(el - hi) + d(fi - ej)][-\gamma_l - \theta b + \theta b\alpha)] + b(hj - fl)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{3,3} &= \{[a(hj - fl) + b(el - hi) + d(fi - ej)][-\gamma_x - \theta c + \theta c\alpha)] + c(hj - fl)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{3,4} &= \{[a(hj - fl) + b(el - hi) + d(fi - ej)][-\gamma_g - \theta d + \theta d\alpha)] + d(hj - fl)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{4,1} &= \{[a(fk - gj) + b(gi - ek) + c(ej - fi)][\gamma_k - \theta(1 - a + a\alpha)] - a(fk - gj)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{4,2} &= \{[a(fk - gj) + b(gi - ek) + c(ej - fi)][-\gamma_l - \theta b + \theta b\alpha)] + b(fk - gj)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{4,3} &= \{[a(fk - gj) + b(gi - ek) + c(ej - fi)][-\gamma_x - \theta c + \theta c\alpha)] + c(fk - gj)(1-\theta)\alpha\}/|D_0| \\
\mathcal{M}_{4,4} &= \{[a(fk - gj) + b(gi - ek) + c(ej - fi)][-\gamma_g - \theta d + \theta d\alpha)] + d(fk - gj)(1-\theta)\alpha\}/|D_0|
\end{aligned}
$$

Starting with the trace of $\mathcal{M}$ multiplied by $|D_0|$ (to simplify the algebra), I find

$$
\begin{aligned}
&\mathrm{trace}(\mathcal{M})|D_0| \\
&\quad = [\gamma_k - \theta(1 - a + \alpha a)]|D_0|/(1-\theta) \\
&\qquad + (1-\theta)\{(hk - gl)(-\alpha b + \gamma_l a) - (ch - dg)i\gamma_l\} \\
&\qquad + \{[a(hj - fl) + b(el - hi) + d(fi - ej)][-\gamma_x - \theta c + \theta c\alpha)] + c(hj - fl)(1-\theta)\alpha\}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \{[a(fk-gj)+b(gi-ek)+c(ej-fi)][-\gamma_g-\theta d+\theta d\alpha)]+d(fk-gj)(1-\theta)\alpha\} \\
&= [\gamma_k-\theta(1-a+\alpha a)][bgl-bhk+chj-dgj] \\
&\quad + (1-\theta)\{(hk-gl)(-\alpha b+\gamma_l a)-(ch-dg)i\gamma_l\} \\
&\quad + \{[a(hj-fl)+b(el-hi)+d(fi-ej)][-\gamma_x-\theta c+\theta c\alpha)]+c(hj-fl)(1-\theta)\alpha\} \\
&\quad + \{[a(fk-gj)+b(gi-ek)+c(ej-fi)][-\gamma_g-\theta d+\theta d\alpha)]+d(fk-gj)(1-\theta)\alpha\} \\
&= [bgl-bhk+chj-dgj]\gamma_k \\
&\quad + [bgl-bhk+chj-dgj]\alpha(1-\theta) \\
&\quad - [bgl-bhk+chj-dgj]\theta \\
&\quad + [a(hk-gl)-i(ch-dg)](1-\theta)\gamma_l \\
&\quad - [a(hj-fl)+b(el-hi)+d(fi-ej)]\gamma_x \\
&\quad - [a(fk-gj)+b(gi-ek)+c(ej-fi)]\gamma_g \\
&= [bgl-bhk+chj-dgj]\gamma_k \\
&\quad [a(hk-gl)-i(ch-dg)](1-\theta)\gamma_l \\
&\quad - [a(hj-fl)+b(el-hi)+d(fi-ej)]\gamma_x \\
&\quad - [a(fk-gj)+b(gi-ek)+c(ej-fi)]\gamma_g \\
&\quad + (\alpha-\theta/(1-\theta))|D_0| \\
&= \left\{\left(\frac{\theta-\phi_{xk}}{\phi_{xk'}(1-\theta)}\right)+\left(\alpha-\frac{\theta}{1-\theta}\right)\right\}|D_0|
\end{aligned}
$$

where the last step involves writing out all terms and cancelling.

Next, I need to show that the determinant of $\mathcal{M}$ is 0. The proof of this is simple because it follows immediately from the fact that the last two rows of D_1 are linearly dependent and proportional to n'. No matter what is the magnitude of the elements of D_0^{-1} or the first two rows of D_1, the determinant must be zero. (Formally, the determinant of the matrix $AB+Cn'$ is 0 where A, B, and C are arbitrary matrices of dimension 4×2, 2×4, and 4×1, respectively.)

Finally, I need to show that $|\mathcal{M} - \alpha I| = 0$. I do so using the fact that $|\mathcal{M}| = 0$ as follows:

$$
\begin{aligned}
&\det(\mathcal{M} - \alpha I) \\
&\quad = |\mathcal{M}| - \alpha\{(\mathcal{M}_{2,2} - \alpha)[(\mathcal{M}_{3,3} - \alpha)(\mathcal{M}_{4,4} - \alpha) - \mathcal{M}_{3,4}\mathcal{M}_{4,3}] \\
&\qquad\qquad - \mathcal{M}_{2,3}[\mathcal{M}_{3,2}(\mathcal{M}_{4,4} - \alpha) - \mathcal{M}_{3,4}\mathcal{M}_{4,2}] \\
&\qquad\qquad + \mathcal{M}_{2,4}[\mathcal{M}_{3,2}\mathcal{M}_{4,3} - \mathcal{M}_{4,2}(\mathcal{M}_{3,3} - \alpha)]\} \\
&\qquad + \mathcal{M}_{1,1}((\mathcal{M}_{2,2} - \alpha)(\alpha^2 - \alpha\mathcal{M}_{3,3} - \alpha\mathcal{M}_{4,4}) + \alpha\mathcal{M}_{2,3}\mathcal{M}_{3,2} + \alpha\mathcal{M}_{2,4}\mathcal{M}_{4,2}) \\
&\qquad - \mathcal{M}_{1,2}(\mathcal{M}_{2,1}(\alpha^2 - \alpha\mathcal{M}_{3,3} - \alpha\mathcal{M}_{4,4}) + \alpha\mathcal{M}_{2,3}\mathcal{M}_{3,1} + \alpha\mathcal{M}_{2,4}\mathcal{M}_{4,1}) \\
&\qquad + \mathcal{M}_{1,3}(-\alpha\mathcal{M}_{2,1}\mathcal{M}_{3,2} + \alpha(\mathcal{M}_{2,2} - \alpha)\mathcal{M}_{3,1}) \\
&\qquad - \mathcal{M}_{1,4}(\alpha\mathcal{M}_{2,1}\mathcal{M}_{4,2} - \alpha(\mathcal{M}_{2,2} - \alpha)\mathcal{M}_{4,1}) \\
&\quad = -\alpha\{\alpha^2 - \alpha\mathcal{M}_{2,2}\mathcal{M}_{3,3} - \alpha\mathcal{M}_{2,2}\mathcal{M}_{4,4} + \mathcal{M}_{2,2}\mathcal{M}_{3,3}\mathcal{M}_{4,4} - \mathcal{M}_{2,2}\mathcal{M}_{3,4}\mathcal{M}_{4,3} \\
&\qquad\qquad - \alpha^3 + \alpha^2\mathcal{M}_{3,3} + \alpha^2\mathcal{M}_{4,4} - \alpha\mathcal{M}_{3,3}\mathcal{M}_{4,4} + \alpha\mathcal{M}_{3,4}\mathcal{M}_{4,3} \\
&\qquad\qquad - \mathcal{M}_{2,3}\mathcal{M}_{3,2}\mathcal{M}_{4,4} + \alpha\mathcal{M}_{2,3}\mathcal{M}_{3,2} + \mathcal{M}_{2,3}\mathcal{M}_{3,4}\mathcal{M}_{4,2} \\
&\qquad\qquad + \mathcal{M}_{2,4}\mathcal{M}_{3,2}\mathcal{M}_{4,3} - \mathcal{M}_{2,4}\mathcal{M}_{4,2}\mathcal{M}_{3,3} + \alpha\mathcal{M}_{2,4}\mathcal{M}_{4,2} \\
&\qquad\qquad - \alpha^2\mathcal{M}_{1,1} + \alpha\mathcal{M}_{1,1}\mathcal{M}_{2,2} + \alpha\mathcal{M}_{1,1}\mathcal{M}_{3,3} + \alpha\mathcal{M}_{1,1}\mathcal{M}_{4,4} - \mathcal{M}_{1,1}\mathcal{M}_{2,2}\mathcal{M}_{3,3} \\
&\qquad\qquad\qquad - \mathcal{M}_{1,1}\mathcal{M}_{2,2}\mathcal{M}_{4,4} + \mathcal{M}_{1,1}\mathcal{M}_{2,3}\mathcal{M}_{3,2} + \mathcal{M}_{1,1}\mathcal{M}_{2,4}\mathcal{M}_{4,2} \\
&\qquad\qquad - \alpha\mathcal{M}_{1,2}\mathcal{M}_{2,1} + \mathcal{M}_{1,2}\mathcal{M}_{2,1}\mathcal{M}_{3,3} + \mathcal{M}_{1,2}\mathcal{M}_{2,1}\mathcal{M}_{4,4} \\
&\qquad\qquad\qquad - \mathcal{M}_{1,2}\mathcal{M}_{2,3}\mathcal{M}_{3,1} - \mathcal{M}_{1,2}\mathcal{M}_{2,4}\mathcal{M}_{4,1} \\
&\qquad\qquad - \alpha\mathcal{M}_{1,3}\mathcal{M}_{3,1} - \mathcal{M}_{1,3}\mathcal{M}_{2,1}\mathcal{M}_{3,2} + \mathcal{M}_{1,3}\mathcal{M}_{2,2}\mathcal{M}_{3,1} \\
&\qquad\qquad - \alpha\mathcal{M}_{1,4}\mathcal{M}_{4,1} - \mathcal{M}_{1,4}\mathcal{M}_{2,1}\mathcal{M}_{4,2} + \mathcal{M}_{1,4}\mathcal{M}_{2,2}\mathcal{M}_{4,1}\} \\
&\quad = -\alpha\{-\alpha^3 + \alpha^2 - \alpha^2\mathcal{M}_{1,1} + \alpha^2\mathcal{M}_{3,3} + \alpha^2\mathcal{M}_{4,4} \\
&\qquad\qquad + \alpha\mathcal{M}_{1,1}\mathcal{M}_{2,2} + \alpha\mathcal{M}_{1,1}\mathcal{M}_{3,3} + \alpha\mathcal{M}_{1,1}\mathcal{M}_{4,4} \\
&\qquad\qquad - \alpha\mathcal{M}_{2,2}\mathcal{M}_{3,3} - \alpha\mathcal{M}_{2,2}\mathcal{M}_{4,4} - \alpha\mathcal{M}_{3,3}\mathcal{M}_{4,4} \\
&\qquad\qquad - \alpha\mathcal{M}_{1,2}\mathcal{M}_{2,1} - \alpha\mathcal{M}_{1,3}\mathcal{M}_{3,1} - \alpha\mathcal{M}_{1,4}\mathcal{M}_{4,1} \\
&\qquad\qquad + \alpha\mathcal{M}_{2,4}\mathcal{M}_{4,2} + \alpha\mathcal{M}_{2,3}\mathcal{M}_{3,2} + \alpha\mathcal{M}_{3,4}\mathcal{M}_{4,3} \\
&\qquad\qquad - \mathcal{M}_{1,1}\mathcal{M}_{2,2}\mathcal{M}_{3,3} - \mathcal{M}_{1,1}\mathcal{M}_{2,2}\mathcal{M}_{4,4} + \mathcal{M}_{2,2}\mathcal{M}_{3,3}\mathcal{M}_{4,4} - \mathcal{M}_{2,2}\mathcal{M}_{3,4}\mathcal{M}_{4,3} \\
&\qquad\qquad + \mathcal{M}_{1,1}\mathcal{M}_{2,3}\mathcal{M}_{3,2} + \mathcal{M}_{1,1}\mathcal{M}_{2,4}\mathcal{M}_{4,2} + \mathcal{M}_{1,2}\mathcal{M}_{2,1}\mathcal{M}_{3,3} + \mathcal{M}_{1,2}\mathcal{M}_{2,1}\mathcal{M}_{4,4} \\
&\qquad\qquad - \mathcal{M}_{1,2}\mathcal{M}_{2,3}\mathcal{M}_{3,1} - \mathcal{M}_{1,2}\mathcal{M}_{2,4}\mathcal{M}_{4,1} - \mathcal{M}_{1,3}\mathcal{M}_{3,2}\mathcal{M}_{2,1} \\
&\qquad\qquad + \mathcal{M}_{1,3}\mathcal{M}_{2,2}\mathcal{M}_{3,1} - \mathcal{M}_{1,4}\mathcal{M}_{4,2}\mathcal{M}_{2,1} + \mathcal{M}_{1,4}\mathcal{M}_{2,2}\mathcal{M}_{4,1} \\
&\qquad\qquad - \mathcal{M}_{2,3}\mathcal{M}_{3,2}\mathcal{M}_{4,4} + \mathcal{M}_{2,3}\mathcal{M}_{3,4}\mathcal{M}_{4,2} + \mathcal{M}_{2,4}\mathcal{M}_{3,2}\mathcal{M}_{4,3} - \mathcal{M}_{2,4}\mathcal{M}_{4,2}\mathcal{M}_{3,3}\} \\
&\quad = 0
\end{aligned}
$$