**Result: Fisher's Example Is Effectively A 1-Shock Example**
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*Claim.* The following economy has *transformed* first-order conditions that are exactly the same as the RBC model in CKM with $\tau_{lt} = 0$ at all dates. (Assume here that population does not grow so that $L_t$ is stationary; it is easy to extend this to have population growth.) The economy is:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log C_t + \psi \log(1 - L_t)\}$$

$$\begin{aligned} \text{subj. to } \; V_t C_t + K_{t+1} - (1-\delta)K_t &= V_t A_t K_t^{\alpha} L_t^{1-\alpha} \\ &= K_t^{\alpha} (Z_t L_t)^{1-\alpha} \end{aligned}$$

where $Z_t^{1-\alpha} = V_t A_t$ and $V_t$ and $A_t$ are unit root processes. Because $V$ and $A$ are unit roots, I can write

$$\begin{aligned} \log Z_t &= (\log V_t + \log A_t)/(1-\alpha) \\ &= (\log V_{t-1} + \log A_{t-1} + \epsilon_{v,t} + \epsilon_{a,t})/(1-\alpha) \\ &= \log Z_{t-1} + (\epsilon_{v,t} + \epsilon_{a,t})/(1-\alpha) \\ &\equiv \log Z_{t-1} + \log z_t \end{aligned}$$

or $Z_t/Z_{t-1} = z_t$, where $z_t = \exp\{(\epsilon_{v,t} + \epsilon_{a,t})/(1-\alpha)\}$. Here, I am ignoring constant terms for convenience because computing equilibria is done with demeaned variables.

*Proof.* The Lagrangian in this case is

$$\mathcal{L} = E_0 \big[ \sum_{t=0}^{\infty} \beta^t \{\log C_t + \psi \log(1 - L_t)\} + \lambda_t (K_t^{\alpha} (Z_t L_t)^{1-\alpha} - V_t C_t - K_{t+1} + (1-\delta) K_t) \big].$$

Taking derivatives, I have the following first-order conditions:

$$\begin{aligned} 1/C_t &= \lambda_t V_t \\ \psi/(1-L_t) &= \lambda_t (1-\alpha) K_t^{\alpha} Z_t^{1-\alpha} L_t^{-\alpha} \\ \lambda_t &= \beta E_t \lambda_{t+1} [1 - \delta + \alpha K_{t+1}^{\alpha-1} (Z_{t+1} L_{t+1})^{1-\alpha}] \end{aligned}$$

which can be rewritten without Lagrange multipliers as follows:

$$\begin{aligned} \psi C_t V_t / (1-L_t) &= (1-\alpha) K_t^{\alpha} Z_t^{1-\alpha} L_t^{-\alpha} \\ 1 &= \beta E_t C_t V_t / (C_{t+1} V_{t+1}) [1 - \delta + \alpha K_{t+1}^{\alpha-1} (Z_{t+1} L_{t+1})^{1-\alpha}]. \end{aligned}$$

Make the following transformations:

$$\hat{c}_t = C_t V_t / Z_t$$
$$\hat{k}_t = K_t / Z_{t-1}$$

and substitute them into the first order conditions:

$$\psi \hat{c}_t = (1-\alpha)\hat{k}_t^{\alpha} z_t^{-\alpha} L_t^{-\alpha}$$

$$1 = \beta E_t (\hat{c}_t / \hat{c}_{t+1}) z_{t+1}^{-1} [1 - \delta + \alpha \hat{k}_{t+1}^{\alpha-1} (z_{t+1} L_{t+1})^{1-\alpha}].$$

$$\hat{c}_t + \hat{k}_{t+1} - (1-\delta)\hat{k}_t z_t^{-1} = \hat{k}_t^{\alpha} z_t^{-\alpha} L_t^{1-\alpha}.$$

This is exactly what CKM have in the no-tax case ($\tau_{lt} = 0$) with no population growth ($g_n = 0$). Therefore, the decision functions for capital and labor can be written in terms of $\hat{k}_t$ and $z_t$. There is no need to keep track of innovations to $V$ and $A$ separately. ■