NOTES: Staff Report 315

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1. Benchmark model

In this section we analyze the maximization problem for a stand-in household with civilians and draftees. Total output in the model is the sum of nonmilitary output and military compensation. Nonmilitary output is produced with civilian hours, private capital, and public capital that does not include military equipment or structures. Private and public capital used to produce nonmilitary output are assumed to be perfect substitutes. Here, we do not distinguish the two margins for adjusting the labor input: hours per worker and the fraction employed. Later we will.

1.1. Household problem

The representative household has two types of family members, civilians and draftees. In period t, fraction $1-a$ are civilians and fraction a are drafted (i.e., in the "army"). Civilians can choose their level of hours but draftees cannot. The problem the household solves is given by

$$\max_{\{c_{ct},c_{dt},i_{pt},l_{ct}\}} E\sum_{t=0}^{\infty}\beta^t\,\left\{(1-a_t)U(c_{ct},l_{ct})+a_tU(c_{dt},\bar{l})\right\}(1+g_p)^t$$

$$\begin{aligned}\text{subject to}\quad & (1-a_t)c_{ct}+a_tc_{dt}+i_t=(1-\tau_{kt})r_tk_{pt}+(1-\tau_{lt})(1-a_t)w_tl_{ct}+\tau_{kt}\delta k_{pt}+T_t\\ & k_{pt+1}=[(1-\delta)k_{pt}+i_{pt}]/(1+g_p)\\ & i_{pt}\geq 0\quad\text{in all states}\\ & c_{ct}\leq c_{max}\quad\text{in some states}\\ & c_{dt}\leq c_{max}\quad\text{in some states}\\ & l_{ct}\leq l_{max}\quad\text{in some states}\end{aligned}$$

with processes for a_t, r_t, w_t, τ_{kt}, τ_{lt}, and T_t given. Quantities are in per-capita terms and the population grows at rate g_p. For now, we assume that civilians choose the total labor input. Later, we will consider choices of both hours per week and employment.

From here on, we will assume that $U(c,l) = \log(c) + V(1-l)$ so that the marginal utilities of civilians and draftees are equated. The Lagrangian for the optimization problem in this case is:

$$\begin{aligned}
\mathcal{L} = E\sum_t \tilde{\beta}^t \Big\{ & (1-a_t)[\log(\hat{c}_{ct}) + V(1-l_{ct})] + a_t \log(\hat{c}_{dt}) \\
& + \frac{\zeta}{3}\big[\min(\hat{i}_{pt},0)^3 + (1-a_t)\min(l_{max}(s_t) - l_{ct},0)^3 \\
& \quad + (1-a_t)\min(c_{max}(s_t) - \hat{c}_{ct},0)^3 + a_t \min(c_{max}(s_t) - \hat{c}_{dt},0)^3\big] \\
& + \mu_t\Big\{(1-\tau_{kt})r_t\hat{k}_{pt} + (1-\tau_{lt})(1-a_t)\hat{w}_t l_{ct} + \tau_{kt}\delta\hat{k}_{pt} + \hat{T}_t \\
& \quad - (1-a_t)\hat{c}_{ct} - a_t\hat{c}_{dt} - \hat{i}_{pt}\Big\} \\
& + \lambda_t\Big\{(1-\delta)\hat{k}_{pt} + \hat{i}_{pt} - (1+g_p)(1+g_z)\hat{k}_{t+1}\Big\}\Big\}
\end{aligned}$$

where $\tilde{\beta} = (1+g_p)\beta$. Variables that grow over time are detrended and denoted with a hat (e.g., $\hat{c}_{ct} = c_{ct}/(1+g_z)^t$). Note that we have added penalty functions to account for our inequality constraints when computing equilibria.

The first-order conditions are thus:

$$1/\hat{c}_{ct} - \zeta\min(c_{max}(s_t) - \hat{c}_{ct},0)^2 = \mu_t$$

$$1/\hat{c}_{dt} - \zeta\min(c_{max}(s_t) - \hat{c}_{dt},0)^2 = \mu_t$$

$$V'(1-l_{ct}) + \zeta\min(l_{max}(s_t) - l_{ct},0)^2 = \mu_t(1-\tau_{lt})\hat{w}_t$$

$$\zeta\min(\hat{i}_{pt},0)^2 + \lambda_t = \mu_t$$

$$(1+g_p)(1+g_z)\lambda_t = \tilde{\beta}E_t\left[\lambda_{t+1}(1-\delta) + \mu_{t+1}\big\{(1-\tau_{kt+1})r_{t+1} + \tau_k\delta\big\}\right].$$

and equilibrium equations for computation can be summarized as follows:

$$\mu_t = 1/\hat{c}_{pt} - \zeta \min(c_{max}(s_t) - \hat{c}_{pt}, 0)^2$$

$$\mu_t - \zeta \min(\hat{i}_{pt}, 0)^2 = \hat{\beta} E_t \left[\mu_{t+1}\{(1-\tau_{kt+1})(r_{t+1}-\delta)+1\} - (1-\delta)\zeta \min(\hat{i}_{pt+1}, 0)^2\right]$$

$$V'(1-l_{ct}) + \zeta \min(l_{max}(s_t) - l_{ct})^2 = \mu_t(1-\tau_{lt})\hat{w}_t$$

where $\hat{\beta} = \beta/(1+g_z)$. Notice that $\hat{c}_{ct} = \hat{c}_{dt}$ in equilibrium. We let $\hat{c}_{pt}$ represent both later.

1.2. Firms

The firm's problem in t is:

$$\max_{\{K_t, L_{pt}\}} \; F(K_t, Z_t L_{pt}) - r_t K_t - w_t L_{pt}$$

where L_p is the total labor input (and equal to the fraction of civilians in the population $(1-a)$ times the ratio of total civilian hours to total civilians (l_c) times the population $((1+g_p)^t)$.

In equilibrium, the rental price and the wage rate are given by:

$$r_t = F_1(K_t, Z_t L_{pt})$$
$$w_t = F_2(K_t, Z_t L_{pt}) Z_t$$

Suppose $F(K,L) = K^\theta L^{1-\theta}$. Then,

$$r_t = \theta K_t^{\theta-1}(Z_t L_{pt})^{1-\theta} = \theta \hat{k}_t^{\theta-1}(z_t l_{ct}(1-a_t))^{1-\theta}$$

$$w_t = (1-\theta) K_t^\theta Z_t^{1-\theta} L_{pt}^{-\theta} = (1+g_z)^t (1-\theta) \hat{k}_t^\theta (z_t l_{ct}(1-a_t))^{-\theta}$$

1.3. The Government

The government's period t budget constraint is given by:

$$C_{gt} + I_{gt} + (1+g_p)^t T_t = \tau_{kt}(r_t - \delta) K_{pt} + \tau_{lt} w_t L_{pt} + r_t K_{gt}$$

where we are assuming that wage payments to soldiers are included in transfers to draftees. Spending is therefore:

$$G_t = C_{gt} + I_{gt} + (1+g_p)^t a_t w_t \bar{l}$$

which when detrended for technological growth is

$$\hat{g}_t = \hat{c}_{gt} + \hat{i}_{gt} + a_t \hat{w}_t \bar{l}.$$

Government capital (which is privately operated) is assumed to have the same rate of depreciation as private capital:

$$K_{gt+1} = (1-\delta)K_{gt} + I_{gt}.$$

When normalized this equation becomes

$$(1+g_p)(1+g_z)\hat{k}_{gt+1} = (1-\delta)\hat{k}_{gt} + \hat{i}_{gt}.$$

1.4. Aggregates

Total private consumption, total private hours (which includes nonmilitary government hours), total private investment, and total private capital are given by:

$$\begin{aligned}
C_{pt} &= (1+g_p)^t[(1-a_t)c_{ct} + a_t c_{dt}] = (1+g_p)^t c_{ct} \\
L_{pt} &= (1+g_p)^t(1-a_t)l_{ct} \\
I_{pt} &= (1+g_p)^t i_{pt} \\
K_{pt} &= (1+g_p)^t k_{pt}
\end{aligned}$$

The resource constraint for this economy is:

$$C_{pt} + I_{pt} + C_{gt} + I_{gt} = F(K_t, Z_t L_{pt})$$

or

$$\hat{c}_{pt} + \hat{i}_{pt} + \hat{c}_{gt} + \hat{i}_{gt} = F(\hat{k}_t, z_t l_{pt}).$$

1.5. Exogenous stochastic processes

The exogenous stochastic processes in this economy are $\{a, \hat{c}_g, \hat{i}_g, \tau_k, \tau_l, z\}$. Let s index the state, where s is determined by a nth-order Markov chain. We assume that at time t if the state is s, then $a_t = a(s)$, $\hat{c}_{gt} = \hat{c}_g(s)$, $\hat{i}_{gt} = \hat{i}_g(s)$, $\tau_{kt} = \tau_k(s)$, $\tau_{lt} = \tau_l(s)$, and $z_t = z(s)$. The process for s is intended to capture different stages of war and/or peace and different levels of technology.

1.6. Steady State Equations

We will use the following functional forms for disutility and production:

$$V(1-l) = \psi[(1-l)^{\xi} - 1]/\xi \tag{1.1}$$

$$F(K, L) = K^{\theta} L^{1-\theta}. \tag{1.2}$$

In this case, the steady state solves

$$\begin{aligned}
r &= [(1+g_z)/\beta - 1]/(1-\tau_k) + \delta \\
\hat{k}_g &= \hat{i}_g/[(1+g_p)(1+g_z) - 1 + \delta] \\
\hat{k}_p &= \hat{i}_p/[(1+g_p)(1+g_z) - 1 + \delta] \\
\hat{y} &= (\hat{k}_p + \hat{k}_g)^{\theta}(z(1-a)l_c)^{1-\theta} \\
\theta &= r(\hat{k}_p + \hat{k}_g)/\hat{y} \\
\hat{c}_p &= \hat{y} - \hat{c}_g - \hat{i}_p - \hat{i}_g \\
\psi &= (1-\tau_l)(1-\theta)(1-l_c)^{1-\xi}\hat{y}/[\hat{c}_p(1-a)l_c]
\end{aligned}$$

which is 7 equations in 7 unknowns (r, $\hat{k}_g$, $\hat{k}_p$, $\hat{y}$, $\hat{c}_p$, θ, ψ) with $\hat{c}_g$, $\hat{i}_g$, a, τ_k, τ_l, z, g_p, g_z, δ, β, ξ, i_p, and l_c given.

1.7. Algorithm for Computing the Consumption Function

When writing the codes, we use x for total capital, i as the index for today's exogenous state, j as the index for tomorrow's exogenous state, I as total number of exogenous states, and π_{ij} as probability of transiting from i to j. Here, we describe out we compute the decision function $\hat{c}_p(x,i)$, which is represented as a weighted sum of known basis functions,

$$\hat{c}_p = \sum_{k=1}^{nnodes} \alpha_k^i N_k(x)$$

where the N_k is a "tent" function that takes on nonzero values in 2 elements on the grid over x surrounding node k, that is

$$N_k(x) = \begin{cases} \frac{x-x_{a-1}}{x_a-x_{a-1}} & x_{a-1} \leq x \leq x_a \\ \frac{x_{a+1}-x}{x_{a+1}-x_a} & x_a \leq x \leq x_{a+1} \\ 0 & \text{elsewhere} \end{cases}$$

We apply a finite element method. This means that we find unknown coefficients α_k, $k = 1, \ldots nnodes$ that satisfy the following equations: $\int R(x,i;\alpha) N_k(x)\, dx = 0$ for all i and k where

$$\begin{aligned} R(x,i;\alpha) &= \mu - \zeta \min(\hat{i}_p, 0)^2 + \hat{\beta}(1-\delta)\zeta \sum_j \pi_{i,j} \min(\hat{i}'_p, 0)^2 \\ &\quad - \hat{\beta} \sum_j \pi_{i,j}\, \mu' \left\{(1-\tau_k(j))[F_1(x', z(j)(1-a(j))l'_c) - \delta] + 1\right\} \end{aligned}$$

and $\alpha = [\alpha_1^1, \ldots, \alpha_{nnodes}^1, \ldots \alpha_{nnodes}^I]'$. The multipliers μ and μ' are

$$\mu = \left(\sum_k \alpha_k^i N_k(x)\right)^{-1} - \zeta \min(c_{max}(i) - \sum_k \alpha_k^i N_k(x), 0)^2$$

$$\mu' = \left(\sum_k \alpha_k^j N_k(x')\right)^{-1} - \zeta \min(c_{max}(j) - \sum_k \alpha_k^j N_k(x'), 0)^2$$

since $\hat{c}_p = \sum_k \alpha_k^i N_k(x)$. The private investments i_p and i'_p satisfy resource constraints:

$$\hat{i}_p = F(x, z(i)(1-a(i))l_c) - \sum_k \alpha_k^i N_k(x) - \hat{c}_g(i) - \hat{i}_g(i)$$

$$\hat{i}'_p = F(x', z(j)(1-a(j))l'_c) - \sum_k \alpha_k^j N_k(x') - \hat{c}_g(j) - \hat{i}_g(j).$$

The next period capital stock is given by:

$$x' = ((1-\delta)x + (\hat{i}_p + \hat{i}_g(i)))/[(1+g_p)(1+g_z)]$$

The labor inputs l_c and l'_c solve:

$$V'(1-l_c) + \zeta \min(l_{max}(i) - l_c, 0)^2 = \mu(1-\tau_l(i))F_2(x, z(i)(1-a(i))l_c)z(i)$$

$$V'(1-l'_c) + \zeta \min(l_{max}(j) - l'_c, 0)^2 = \mu'(1-\tau_l(j))F_2(x', z(j)(1-a(j))l'_c)z(j).$$

Derivatives of the residual equation are as follows:

$$\begin{aligned}
\frac{\partial R(x,i;\alpha)}{\partial \alpha_k^i} &= \left(\frac{\partial \mu}{\partial \hat{c}_p} - 2\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}_p}{d\hat{c}_p}\right)\frac{d\hat{c}_p}{\alpha_k^i} \\
&\quad + \hat{\beta}\sum_j \pi_{i,j}\ \{(1-\tau_k(j))[F_1(x', z(j)(1-a(j))l'_c) - \delta] + 1\}\ \frac{d\mu'}{d\hat{c}'_p}\frac{d\hat{c}'_p}{d\alpha_k^i} \\
&\quad - \hat{\beta}\sum_j \pi_{i,j}\, \mu'(1-\tau_k(j)) \\
&\qquad \Bigg(F_{11}(x', z(j)(1-a(j))l'_c)\frac{dx'}{d\alpha_k^i} \\
&\qquad + F_{12}(x', z(j)(1-a(j))l'_c)z(j)(1-a(j))\frac{dl'_c}{d\alpha_k^i}\Bigg) \\
&\quad + \hat{\beta}(1-\delta)\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}'_p}{d\alpha_k^i}
\end{aligned}$$

and for $j \neq i$

$$\begin{aligned}
\frac{\partial R(x,i;\alpha)}{\partial \alpha_k^j} &= \hat{\beta}\pi_{i,j}\ \{(1-\tau_k(j))[F_1(x', z(j)(1-a(j))l'_c) - \delta] + 1\}\ d\mu' d\hat{c}'_p \frac{d\hat{c}'_p}{d\alpha_k^j} \\
&\quad - \hat{\beta}\pi_{i,j}\, \mu'(1-\tau_k(j))F_{12}(x', z(j)(1-a(j))l'_c)z(j)(1-a(j))\frac{dl'_c}{d\alpha_k^j} \\
&\quad + \hat{\beta}(1-\delta)\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}'_p}{d\alpha_k^j}
\end{aligned}$$

The partial derivative of μ is

$$\frac{d\mu}{d\hat{c}_p} = -\frac{1}{\hat{c}_p^2} + 2\zeta \min(c_{max}(i) - \hat{c}_p, 0)$$

For the partial derivatives of l_c and l_c', totally differentiate the static first order condition for the consumer to get:

$$\begin{aligned} &- \left[V''(1-l_c) + 2\zeta \min(l_{max}(i) - l_c, 0)\right] dl_c \\ &\qquad = d\mu(1-\tau_l)F_2 z + \mu(1-\tau_l)\left[F_{12}dx + F_{22}z(1-a)dl_c\right] z(1-a) \end{aligned}$$

where the arguments of F are $(x, z(1-a)l_c)$. For the current period l_c, $dx = 0$ since x is a given state variable. For l_c', dx' is given by:

$$\begin{aligned} dx' &= d\hat{i}_p/[(1+g_p)(1+g_z)] \\ &= (F_2(x, z(1-a)l_c)z(1-a)dl_c - d\hat{c}_p)/[(1+g_p)(1+g_z)] \end{aligned}$$

Therefore, we have

$$\frac{dl_c}{d\hat{c}_p} = -\frac{(1-\tau_l)F_2 z}{V'' + 2\zeta \min(l_{max}(i) - l_c, 0) + \mu(1-\tau_l)F_{22}z^2(1-a)^2} \frac{d\mu}{d\hat{c}_p}$$

where the argument of V is $1 - l_c$ and the arguments of F are $(x, z(i)(1-a(i))l_c)$, and in the next period we have

$$dl_c' = \frac{V'\, d\hat{c}_p' - (1-\tau_l(j))F_{12}z(j)(1-a(j))\, dx'}{V''\hat{c}_p' + (1-\tau_l(j)F_{22}z(j)^2(1-a(j))^2}$$

where the argument of V is $1 - l_c'$ and the arguments of F are $(x', z(j)(1-a(j))l_c')$. Finally we need

$$\frac{d\hat{c}_p'}{d\alpha_k^i} = \left(\sum_l \alpha_l^j \frac{\partial N_k(x')}{\partial x'}\right) \frac{dx'}{d\alpha_k^i}$$

$$\frac{d\hat{c}_p'}{d\alpha_k^j} = N_k(x')$$

$$\begin{aligned} \frac{dx'}{d\alpha_k^i} &= \frac{1}{(1+g_p)(1+g_z)} \left(F_2(x, z(1-a)l_c)z(1-a)\frac{dl_c}{d\alpha_k^i} - \frac{d\hat{c}_p}{d\alpha_k^i}\right) \\ &= \frac{1}{(1+g_p)(1+g_z)} \left(F_2(x, z(1-a)l_c)z(1-a)\frac{dl_c}{d\hat{c}_p} - 1\right) \frac{d\hat{c}_p}{d\alpha_k^i} \\ &= \frac{1}{(1+g_p)(1+g_z)} \left(F_2(x, z(1-a)l_c)z(1-a)\frac{dl_c}{d\hat{c}_p} - 1\right) N_k(x) \end{aligned}$$

$$d\hat{i}_p' = F_1(x', z(j)(1-a(j))l_c')\, dx' + F_2(x', z(j)(1-a(j))l_c')z(j)(1-a(j))\, dl_c' - d\hat{c}_p'.$$

2. A Version of the Model with Capacity Utilization

In this section, we consider the extension of the benchmark model considered by Braun and McGrattan (1993) in their Appendix B.

2.1. Household Problem

The maximization problem of the stand-in household is now:

$$\begin{aligned}\max_{\{c_{1t},c_{0t},c_{dt},i_{pt},n_t,h_t\}} E\sum_{t=0}^{\infty}\beta^t \big((1-a_t)\{n_t[\log c_{1t}+V(1-h_t)]& \\ +(1-n_t)[\log c_{0t}+V(1)]-p(n_t)\}& \\ +a_t[\log c_{dt}+V(1-\bar{h})]\big)(1+g_p)^t&\end{aligned}$$

$$\begin{aligned}\text{subject to}\quad &(1-a_t)\{n_t c_{1t}+(1-n_t)c_{0t}\}+a_t c_{dt}+i_{pt}\\ &\qquad =(1-\tau_{kt})r_t k_{pt}+(1-\tau_{lt})(1-a_t)w_t(h_t)n_t\\ &\qquad\qquad +\tau_{kt}\delta k_{pt}+T_t\\ &k_{pt+1}=[(1-\delta)k_{pt}+i_{pt}]/(1+g_p)\\ &i_{pt}\geq 0\quad \text{in all states}\\ &c_{1t}\leq c_{max}\quad \text{in some states}\\ &c_{0t}\leq c_{max}\quad \text{in some states}\\ &c_{dt}\leq c_{max}\quad \text{in some states}\\ &n_t\leq n_{max}\quad \text{in some states}\end{aligned}$$

with processes for a_t, r_{pt}, $w_t(h_t)$, τ_{kt}, τ_{lt}, and T_t given. Quantities are in per-capita terms. The household chooses consumption of those employed c_{1t}, consumption of those not employed c_{0t}, the consumption of the draftees c_{dt}, the number (or fraction) of those employed n_t, the length of the workweek h_t, and investment i_{pt}.[1] Each term in the budget constraint is in per-capita terms. For example, the first term $(1-a_t)n_t c_{1t}$ is

$$\frac{\text{number of civilians}}{\text{total population}}\times\frac{\text{number of civilian workers}}{\text{number of civilians}}\times\frac{\text{consumption of civilian workers}}{\text{number of civilian workers}}.$$

[1] Note that we could assume that c_{dt} is determined by the government as Braun and McGrattan (1993) do. Since we work with log preferences, whether it is given or not will not affect our results.

The function $p(n)$ represents costs—in utility terms—of increasing employment or changing its level. If $p(n) = 0$ then the equilibrium workweek is constant. (See Braun and McGrattan (1993).) The disutility of employment captured by $p(n_t)$ ensures that both the intensive margin and the extensive margin are used. The disutility of employment term was motivated by Braun and McGrattan (1993) as follows. They assume that individual preferences are given by

$$E \sum_{t=0}^{\infty} \beta^t [U(c_t, 1 - h_t) - \eta \chi_{\{h_t > 0\}}]$$

where η measures the disutility of entering the work force and χ is an indicator function. If the utility costs of entering the workforce vary, then η will have a nondegenerate distribution. If civilians are aligned with points on the interval [0,1-a), then we can construct a cost function. For example, suppose that individuals are aligned so that costs are linear and increasing. Then, in the aggregate the costs of increasing employment are given by

$$-(1 - a) \int_0^n (\zeta_0 + 2\zeta_1 x) dx = -(1 - a)(\zeta_0 n + \zeta_1 n^2)$$

with $\zeta_1 > 0$.

The Lagrangian for the optimization problem in this case is:

$$\begin{aligned}
\mathcal{L} = E \sum_t \tilde{\beta}^t \Big\{ & (1 - a_t)\{n_t[\log(\hat{c}_{1t}) + V(1 - h_t)] + (1 - n_t)[\log \hat{c}_{0t} + V(1)]\} \\
& + a_t \log(\hat{c}_{dt}) - (1 - a_t) p(n_t) \\
& + \frac{\zeta}{3} \big[\min(\hat{i}_{pt}, 0)^3 + (1 - a_t) \min(n_{max}(s_t) - n_t, 0)^3 \\
& \quad + (1 - a_t) n_t \min(c_{max}(s_t) - \hat{c}_{1t}, 0)^3 \\
& \quad + (1 - a_t)(1 - n_t) \min(c_{max}(s_t) - \hat{c}_{0t}, 0)^3 \\
& \quad + a_t \min(c_{max}(s_t) - \hat{c}_{dt}, 0)^3\big] \\
& + \mu_t \Big\{ (1 - \tau_{kt}) r_t \hat{k}_{pt} + (1 - \tau_{lt})(1 - a_t) \hat{w}_t(h_t) n_t + \tau_{kt} \delta \hat{k}_{pt} + \hat{T}_t \\
& \quad - (1 - a_t)\{n_t \hat{c}_{1t} - (1 - n_t) c_{0t}\} - a_t \hat{c}_{dt} - \hat{i}_{pt} \Big\} \\
& + \lambda_t \Big\{ (1 - \delta) \hat{k}_{pt} + \hat{i}_{pt} - (1 + g_p)(1 + g_z) \hat{k}_{t+1} \Big\} \Big\}
\end{aligned}$$

where $\tilde{\beta} = (1+g_p)\beta$. As before, variables that grow over time are detrended and denoted with a hat (e.g., $\hat{c}_{ct} = c_{ct}/(1+g_z)^t$).

The first-order conditions are thus:

$$1/\hat{c}_{1t} - \zeta \min(c_{max}(s_t) - \hat{c}_{1t}, 0)^2 = \mu_t$$

$$1/\hat{c}_{0t} - \zeta \min(c_{max}(s_t) - \hat{c}_{0t}, 0)^2 = \mu_t$$

$$1/\hat{c}_{dt} - \zeta \min(c_{max}(s_t) - \hat{c}_{dt}, 0)^2 = \mu_t$$

$$V'(1-h_t) = \mu_t(1-\tau_{lt})\hat{w}_t'(h_t)$$

$$\begin{aligned} 0 = \log \hat{c}_{1t} + V(1-h_t) - \log \hat{c}_{0t} - V(1) - p'(n_t) - \zeta \min(n_{max}(s_t) - n_t, 0)^2 \\ + \zeta \min(c_{max}(s_t) - \hat{c}_{1t}, 0)^2 - \zeta \min(c_{max}(s_t) - \hat{c}_{0t}, 0)^2 \\ + \mu_t[(1-\tau_{lt})\hat{w}_t(h_t) - \hat{c}_{1t} + \hat{c}_{0t}] \end{aligned}$$

$$\zeta \min(\hat{i}_{pt}, 0)^2 + \lambda_t = \mu_t$$

$$(1+g_p)(1+g_z)\lambda_t = \tilde{\beta} E_t \left[\lambda_{t+1}(1-\delta) + \mu_{t+1}\{(1-\tau_{kt+1})r_{t+1} + \tau_k \delta\}\right].$$

and equilibrium equations for computation can be summarized as follows:

$$\mu_t = 1/\hat{c}_{pt} - \zeta \min(c_{max}(s_t) - \hat{c}_{pt}, 0)^2$$

$$\mu_t - \zeta \min(\hat{i}_{pt}, 0)^2 = \hat{\beta} E_t \left[\mu_{t+1}\{(1-\tau_{kt+1})(r_{t+1} - \delta) + 1\} - (1-\delta)\zeta \min(\hat{i}_{pt+1}, 0)^2\right]$$

$$V'(1-h_t) = \mu_t(1-\tau_{lt})\hat{w}_t'(h_t) \tag{2.1}$$

$$V(1-h_t) - V(1) + (1-\theta)V'(1-h_t)h_t/\phi = p'(n_t) + \zeta \min(n_{max}(s_t) - n_t, 0)^2 \tag{2.2}$$

where $\hat{\beta} = \beta/(1+g_z)$ and $\hat{c}_{pt} = \hat{c}_{1t} = \hat{c}_{0t} = \hat{c}_{dt}$.

2.2. Firms

Let's turn next to the production technologies. There are different technologies each defined by length of the workweek, h,

$$Y_t = Z_t^{1-\theta} K_t^{\theta} N_t^{1-\theta} h_t^{\phi} = z_t^{1-\theta} \hat{k}_t^{\theta} (n_t(1-a_t))^{1-\theta} h_t^{\phi}. \tag{2.3}$$

A firm of type h solves the following maximization problem

$$\max_{K_t, N_t} Y_t - r_t K_t - w_t(h_t) N_t$$

subject to (2.3). The rental rate and wage rate is therefore given by

$$\begin{aligned} r_t &= \theta Z_t^{1-\theta} K_t^{\theta-1} N_t^{1-\theta} h_t^{\phi} \\ w_t(h_t) &= (1-\theta) Z_t^{1-\theta} K_t^{\theta} N_t^{-\theta} h_t^{\phi} \end{aligned}$$

The total capital stock K_t and workforce N_t is

$$\begin{aligned} K_t &= (1+g_p)^t (1+g_z)^t \hat{k}_t \\ N_t &= (1+g_p)^t n_t (1-a_t). \end{aligned}$$

Thus, when we normalize rental rates and wage rates, we have

$$\begin{aligned} r_t &= \theta \hat{k}_t^{\theta-1} (z_t n_t (1-a_t))^{1-\theta} h_t^{\phi} \\ \hat{w}_t(h_t) &= (1-\theta) \hat{k}_t^{\theta} z_t^{1-\theta} (n_t(1-a_t))^{-\theta} h_t^{\phi} = w_t(h_t)(1+g_z)^t \\ \hat{w}_t'(h_t) &= \phi \hat{k}_t^{\theta} z_t^{1-\theta} (n_t(1-a_t))^{-\theta} h_t^{\phi-1} \\ &= \phi \hat{w}_t(h_t) / [(1-\theta) h_t] \end{aligned}$$

From the perspective of the household, $\hat{w}_t$ is a function of both h_t which they are choosing and r_t which they are not. To see this, note that

$$\begin{aligned} \hat{w}_t(h_t; r_t) &= (1-\theta) \frac{\hat{k}_t}{z_t n_t (1-a_t)}^{\theta} z_t h_t^{\phi} \\ &= (1-\theta) \frac{r_t}{\theta h_t^{\phi}}^{\frac{\theta}{\theta-1}} z_t h_t^{\phi} \\ &= z_t (1-\theta) \theta^{\frac{\theta}{1-\theta}} r_t^{\frac{\theta}{\theta-1}} h_t^{\frac{\phi}{1-\theta}-1} \end{aligned}$$

2.3. The Government

The government's period t budget constraint is given by:

$$C_{gt} + I_{gt} + (1+g_p)^t T_t = \tau_{kt}(r_t - \delta)K_{pt} + \tau_{lt} w_t(h_t) N_t + r_t K_{gt}$$

where we are assuming that wage payments to soldiers are included in transfers to draftees. Spending is therefore:

$$G_t = C_{gt} + I_{gt} + \text{military compensation.}$$

2.4. Aggregates

Total private consumption, total private labor input (which includes nonmilitary government), total private investment, and total private capital are given by:

$$\begin{aligned} C_{pt} &= (1+g_p)^t[(1-a_t)(n_t c_{1t} + (1-n_t)c_{0t}) + a_t c_{dt}] \\ I_{pt} &= (1+g_p)^t i_{pt} \\ K_{pt} &= (1+g_p)^t k_{pt} \end{aligned}$$

The resource constraint for this economy is:

$$C_{pt} + I_{pt} + C_{gt} + I_{gt} = Z_t^{1-\theta} K_t^{\theta} N_t^{1-\theta} h_t^{\phi}$$

or

$$\hat{c}_{pt} + \hat{i}_{pt} + \hat{c}_{gt} + \hat{i}_{gt} = \hat{k}_t^{\theta}(z_t n_t(1-a_t))^{1-\theta} h_t^{\phi}$$

2.5. Steady State Equations

We will use the same functional form for V and F as before (see (1.1) and (1.2)), and we will try various functional forms for $p(n)$. At the start, we assume

$$p(n) = \eta[n^{\rho} - 1]/\rho.$$

The steady state for this problem is

$$
\begin{aligned}
r &= [(1+g_z)/\beta - 1]/(1-\tau_k) + \delta \\
\hat{k}_g &= \hat{i}_g/[(1+g_p)(1+g_z) - 1 + \delta] \\
\hat{k}_p &= \hat{i}_p/[(1+g_p)(1+g_z) - 1 + \delta] \\
\hat{y} &= (\hat{k}_p + \hat{k}_g)^\theta (z(1-a)n)^{1-\theta} h^\phi \\
\hat{c}_p &= \hat{y} - \hat{c}_g - \hat{i}_p - \hat{i}_g \\
\theta &= r(\hat{k}_p + \hat{k}_g)/\hat{y} \\
\hat{w} &= (1-\theta)\hat{y}/(n(1-a)) \\
\psi &= (1-\tau_l)\phi(1-h)^{1-\xi}\hat{w}/[\hat{c}_p h] \\
\eta &= (\psi((1-h)^\xi - 1)/\xi + (1-\tau_l)\hat{w}/\hat{c}_p)/n^{\rho-1}
\end{aligned}
$$

which is 9 equations in 9 unknowns (r, $\hat{k}_g$, $\hat{k}_p$, $\hat{y}$, $\hat{c}_p$, θ, $\hat{w}$, ψ, η) with $\hat{c}_g$, $\hat{i}_g$, a, τ_k, τ_l, z, g_p, g_z, δ, β, ϕ, ξ, ρ, i_p, h, and n given.

2.6. Algorithm for Computing the Consumption Function

As before, we are computing $c_p(x, i)$, which is represented as a weighted sum of known basis functions,

$$
\hat{c}_p = \sum_{k=1}^{nnodes} \alpha_k^i N_k(x)
$$

with known basis functions. The residual in this case is given by:

$$
\begin{aligned}
R(x, i; \alpha) = \mu - \zeta \min(\hat{i}_p, 0)^2 + \hat{\beta}(1-\delta)\zeta \sum_j \pi_{i,j} \min(\hat{i}'_p, 0)^2 \\
- \hat{\beta} \sum_j \pi_{i,j}\, \mu' \left\{ (1-\tau_k(j))[\theta(x')^{\theta-1}(z(j)(1-a(j))n')^{1-\theta}(h')^\phi - \delta] + 1 \right\}
\end{aligned}
$$

and $\alpha = [\alpha_1^1, \ldots, \alpha_{nnodes}^1, \ldots \alpha_{nnodes}^I]'$. The multipliers μ and μ' are

$$\mu = \left(\sum_k \alpha_k^i N_k(x)\right)^{-1} - \zeta \min(c_{max}(i) - \sum_k \alpha_k^i N_k(x), 0)^2$$

$$\mu' = \left(\sum_k \alpha_k^j N_k(x')\right)^{-1} - \zeta \min(c_{max}(j) - \sum_k \alpha_k^j N_k(x'), 0)^2$$

since $\hat{c}_p = \sum_k \alpha_k^i N_k(x)$. The private investments i_p and i_p' satisfy resource constraints:

$$\hat{i}_p = x^\theta (z(i)(1-a(i))n)^{1-\theta} h^\phi - \sum_k \alpha_k^i N_k(x) - \hat{c}_g(i) - \hat{i}_g(i)$$

$$\hat{i}_p' = (x')^\theta (z(j)(1-a(j))n')^{1-\theta} (h')^\phi - \sum_k \alpha_k^j N_k(x') - \hat{c}_g(j) - \hat{i}_g(j).$$

The next period capital stock is given by:

$$x' = ((1-\delta)x + (\hat{i}_p + \hat{i}_g(i)))/[(1+g_p)(1+g_z)]$$

If we have μ, x, and the exogenous variables, then we can solve the following two equations for the two unknowns, hours of work h and the employment level n:

$$V'(1-h) = \mu(1-\tau_l(i))\phi(1-\theta)(x/n)^\theta z(i)^{1-\theta}(1-a(i))^{-\theta} h^{\phi-1}$$

$$0 = V(1-h) + V'(1-h)h/\phi - p'(n) - \zeta \min(n_{max}(i) - n, 0)^2.$$

Derivatives of the residual equation are given as follows:

$$\frac{\partial R(x,i;\alpha)}{\partial \alpha_k^i} = \left(\frac{\partial \mu}{\partial \hat{c}_p} - 2\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}_p}{d\hat{c}_p}\right)\frac{d\hat{c}_p}{\alpha_k^i}$$

$$+ \hat{\beta}\sum_j \pi_{i,j} \left\{(1-\tau_k(j))[\theta(x')^{\theta-1}(z(j)(1-a(j))n')^{1-\theta}(h')^{\phi} - \delta] + 1\right\} \frac{d\mu'}{d\hat{c}_p'}\frac{d\hat{c}_p'}{d\alpha_k^i}$$

$$- \hat{\beta}\sum_j \pi_{i,j}\, \mu'(1-\tau_k(j))\theta$$

$$\Bigg((\theta-1)(x')^{\theta-2}(z(j)(1-a(j))n')^{1-\theta}(h')^{\phi}\frac{dx'}{d\alpha_k^i}$$

$$+ (1-\theta)(x')^{\theta-1}(z(j)(1-a(j)))^{1-\theta}(n')^{-\theta}(h')^{\phi}\frac{dn'}{d\alpha_k^i}\Bigg)$$

$$+ \phi(x')^{\theta-1}(z(j)(1-a(j))n')^{1-\theta}(h')^{\phi-1}\frac{dh'}{d\alpha_k^i}\Bigg)$$

$$+ \hat{\beta}(1-\delta)\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}_p'}{d\alpha_k^i}$$

and for $j \neq i$

$$\frac{\partial R(x,i;\alpha)}{\partial \alpha_k^j} = \hat{\beta}\pi_{i,j} \left\{(1-\tau_k(j))[\theta(x')^{\theta-1}(z(j)(1-a(j))n')^{1-\theta}(h')^{\phi} - \delta] + 1\right\} d\mu' d\hat{c}_p' \frac{d\hat{c}_p'}{d\alpha_k^j}$$

$$- \hat{\beta}\pi_{i,j}\, \mu'(1-\tau_k(j))\theta(1-\theta)(x')^{\theta-1}(z(j)(1-a(j)))^{1-\theta}(n')^{-\theta}(h')^{\phi}\frac{dn'}{d\alpha_k^j}$$

$$- \hat{\beta}\pi_{i,j}\, \mu'(1-\tau_k(j))\theta\phi(x')^{\theta-1}(z(j)(1-a(j))n')^{1-\theta}(h')^{\phi-1}\frac{dh'}{d\alpha_k^j}$$

$$+ \hat{\beta}(1-\delta)\zeta \min(\hat{i}_p, 0)\frac{d\hat{i}_p'}{d\alpha_k^j}$$

The partial derivative of μ is

$$\frac{d\mu}{d\hat{c}_p} = -\frac{1}{\hat{c}_p^2} + 2\zeta \min(c_{max}(i) - \hat{c}_p, 0)$$

For the partial derivatives of n, n', h, and h', totally differentiate the static consumer's first order conditions (2.1) and (2.2) to get:

$$
\begin{aligned}
&-V''(1-h)dh = V'(1-h)(d\mu/\mu + \theta dx/x - \theta dn/n + (\phi-1)dh/h) \\
&-V'(1-h)dh + (1-\theta)V'(1-h)/\phi dh - (1-theta)V''(1-h)h/\phi dh \\
&\qquad = p''(n)dn - 2\zeta \min(n_{max}(i) - n, 0)dn
\end{aligned}
$$

where $\kappa = (1-\tau_l)\phi z^{1-\theta}(1-a)^{-\theta}$. For the current period, $dx = 0$ since x is a given state variable. For the next period, dx' is given by:

$$
\begin{aligned}
dx' &= d\hat{i}_p/[(1+g_p)(1+g_z)] \\
&= (x^\theta(z(1-a)n)^{1-\theta}h^\phi)((1-\theta)dn/n + \phi dh/h) - d\hat{c}_p)/[(1+g_p)(1+g_z)]
\end{aligned}
$$

Finally we need

$$
\frac{d\hat{c}_p'}{d\alpha_k^i} = \left(\sum_l \alpha_l^j \frac{\partial N_k(x')}{\partial x'}\right) \frac{dx'}{d\alpha_k^i}
$$

$$
\frac{d\hat{c}_p'}{d\alpha_k^j} = N_k(x')
$$

$$
\frac{dx'}{d\alpha_k^i} = \frac{1}{(1+g_p)(1+g_z)} \left((x^\theta(z(1-a)n)^{1-\theta}h^\phi \left[\frac{(1-\theta)}{n}\frac{dn}{d\alpha_k^i} + \frac{\phi}{h}\frac{dh}{d\alpha_k^i}\right] - \frac{d\hat{c}_p}{d\alpha_k^i} \right)
$$

$$
d\hat{i}_p' = (x')^\theta(z(j)(1-a(j))n')^{1-\theta}(h')^\phi \; [\theta dx'/x' + (1-\theta)dn'/n' + \phi dh'/h'] - d\hat{c}_p'.
$$