Finite-Difference Methods for Dynamic Economies

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Introduction

- "Algorithms based on finite-difference approximations encounter fundamental difficulties." Robert Wilson, *Handbook of Computational Economics, Vol. 1,* 1996.

- However, algorithms based on finite-difference approximations are used throughout computational physics and engineering to solve many different problems.
- Can we apply some of these methods to problems in economics?

- In these lectures, we will consider several dynamic economies and discuss how to solve them using finite-difference methods.

- We will start with a simple approach, and see how that works. Then we will use models equations and simple theory to improve the methods.

- A lot more work needs to be done to tailor the methods for these problems.

Stochastic Dynamic Programming

- Agent's Problem:

$$\max_c E \int_0^\infty e^{-\rho t}\, u(c)\, dt$$
$$\text{s.t.}\quad dk = \big(f(k) - c\big)\, dt + \sigma(k)\, dz, \quad k(0) \text{ given}$$

where:

$$c = \text{consumption}$$
$$k = \text{capital stock}$$
$$z = \text{Brownian motion}$$

- The general dynamic programming equation for the value function, $V(k)$, is:

$$V_t + \rho V = \max_c \Big\{ u(c) + V_k\big(f(k) - c\big) + \tfrac{1}{2}\sigma^2 V_{kk} \Big\}$$

let:

$$u(c) = c^\omega/\omega$$
$$f(k) = \lambda k^\theta - \delta k$$
$$\sigma(k) = \sigma k$$

- The first-order condition, $u'(c) = V_k = c^{\omega-1}$, gives the differential equation:

$$V_t + \rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} + V_k\big(\lambda k^\theta - \delta k\big) + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

Exact Solutions for $\sigma = 0$

- Steady-state solution ($k = k_s$):
  - solve first-order conditions of original problem:

$$\begin{aligned} f'(k_s) &= \rho \\ \theta \lambda k_s^{\theta-1} - \delta &= \rho \end{aligned}$$

  - obtain:

$$\begin{aligned} k_s &= \left(\frac{\rho + \delta}{\lambda \theta} \right)^{1/\theta - 1} \\ c_s &= \lambda k_s^{\theta} - \delta k_s \end{aligned}$$

- Special case where $\theta = \omega = \frac{1}{2}$:

$$\begin{aligned} V &= \frac{\phi^{\omega-1}}{\omega} k^{\omega} + \frac{\lambda}{\rho} \phi^{\omega-1} \\ c &= V_k^{1/\omega-1} = \phi k \\ \phi &= \frac{\rho + \delta \omega}{1 - \omega} \end{aligned}$$

Approach to Computing $V(k)$

- Find time-independent solution: $V_t = 0$
- Either solve the functional equation:

$$\rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} + V_k\big(\lambda k^\theta - \delta k\big) + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

  subject to boundary conditions at $k = 0$ and $k = k_{max}$ (BVP)

- Or solve the time-dependent PDE:

$$V_t + \rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} + V_k\big(\lambda k^\theta - \delta k\big) + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

  for the time-independent solution, with an initial condition and subject to boundary conditions (IBVP)

  - use (guessed) initial condition and integrate in time until solution is no longer a function of the initial condition

Approach to Computing $V(k)$

- Time-dependent approach is better:
  - easier to generalize to multiple states
  - can use theory of PDEs and well-developed numerical methods
  - can use size of time step to regulate the progression of the solution
- Many methods can be used for the solution of this problem:
  - finite-difference, finite-volume, finite-element methods

Finite-Difference Methods

- Approximate the function $V(t,k)$ on an equispaced time / capital-stock grid:

$$V(t,k) = V(t_n, k_i) = V(n\,\Delta t, i\,\Delta k) = V_i^n$$

where

$$n = 0, 1, 2, \ldots \qquad i = 0, 1, 2, \ldots, i_{max}$$

- Use finite-difference approximations to represent the derivatives:

$$V_k(t,k) = \frac{\partial V}{\partial k} = \lim_{\Delta k \to 0} \frac{V(t, k+\Delta k) - V(t,k)}{\Delta k}$$

- Use a Taylor series approximation for $V(t, k+\Delta k)$:

$$\begin{aligned}
V(t, k+\Delta k) &= V(t,k) + \Delta k\, V_k(t,k) + \tfrac{1}{2}\Delta k^2\, V_{kk}(t,k) + \ldots \\
\frac{V(t,k+\Delta k) - V(t,k)}{\Delta k} &= V_k(t,k) + \tfrac{1}{2}\Delta k\, V_{kk}(t,k) + \ldots \\
&= V_k(t,k) + O(\Delta k) \\
\frac{V_{i+1}^n - V_i^n}{\Delta k} &= (V_k)_i^n + O(\Delta k)
\end{aligned}$$

- This is a first-order accurate forward-difference approximation to V_k

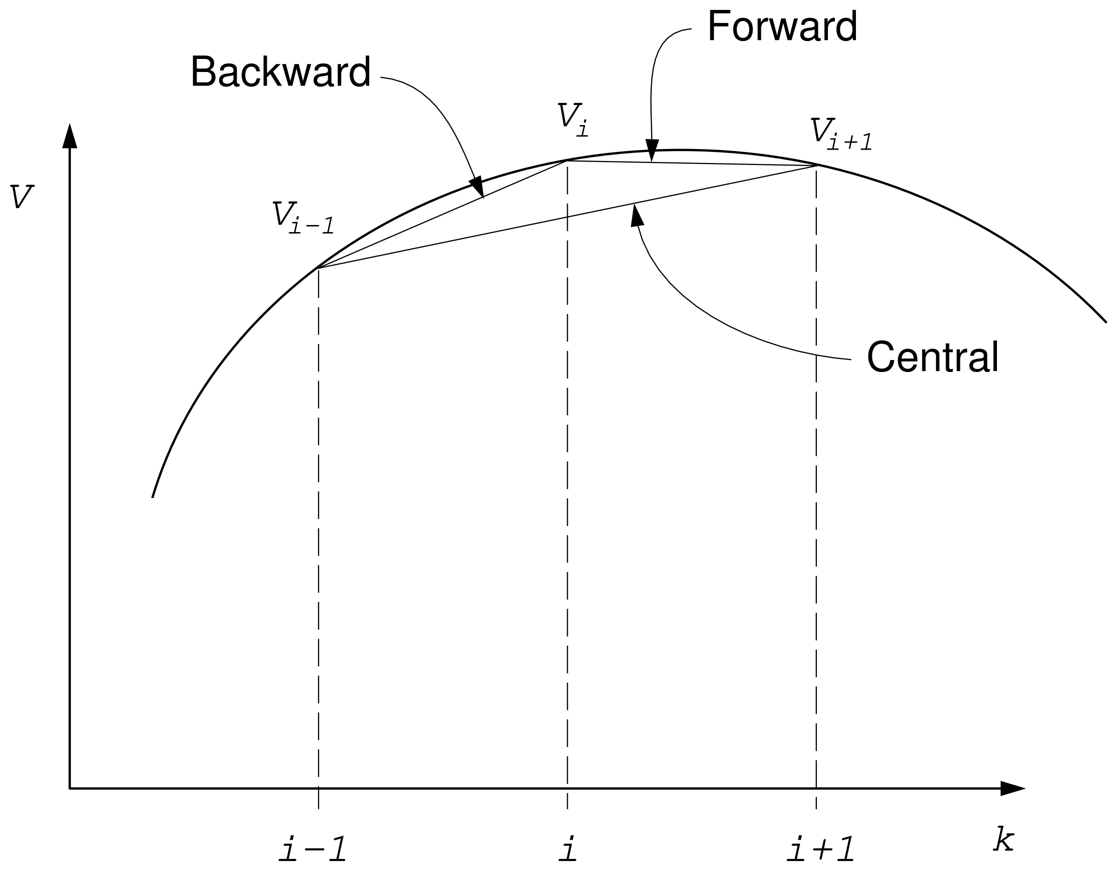
- Similarly, we can derive:

$$(V_k)_i^n = \frac{V_i^n - V_{i-1}^n}{\Delta k} + O(\Delta k) \qquad \text{backward difference}$$
$$(V_k)_i^n = \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta k} + O(\Delta k^2) \qquad \text{central difference}$$
$$(V_k)_i^n = \frac{V_{i+1}^n - V_i^n}{\Delta k} + O(\Delta k) \qquad \text{forward difference}$$

- We can derive approximations of arbitrary order of accuracy by taking linear combinations of Taylor series expansions of $V_{i\pm m}$
  - the more points used, the higher the accuracy
  - central-differences are more accurate than biased approximations
- We can also derive higher-order derivatives. For example

$$(V_{kk})_i^n = \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta k^2} + O(\Delta k^2) \qquad \text{central difference}$$

Finite-Difference Approximations to $(V_k)_i$



- For this case, the central difference approximation is more accurate than the first-order, one-sided differences.

Finite-Difference Solution of $V(k)$

- Time-dependent differential equation:

$$V_t + \rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} + V_k\left(\lambda k^\theta - \delta k\right) + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

- Use a backward-difference approximation for V_t (for ease of solution; more later):

$$(V_t)_i^n = \frac{V_i^{n+1} - V_i^n}{\Delta t} + O(\Delta t)$$

- Use central-difference approximations for V_k and V_{kk}:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + \rho V_i^n = \frac{1-\omega}{\omega}\left(\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta k}\right)^{\omega/\omega-1} + \left(\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta k}\right)\left(\lambda k_i^\theta - \delta k_i\right) + \tfrac{1}{2}\sigma^2 k_i^2 \left(\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta k^2}\right) + O(\Delta t) + O(\Delta k^2)$$

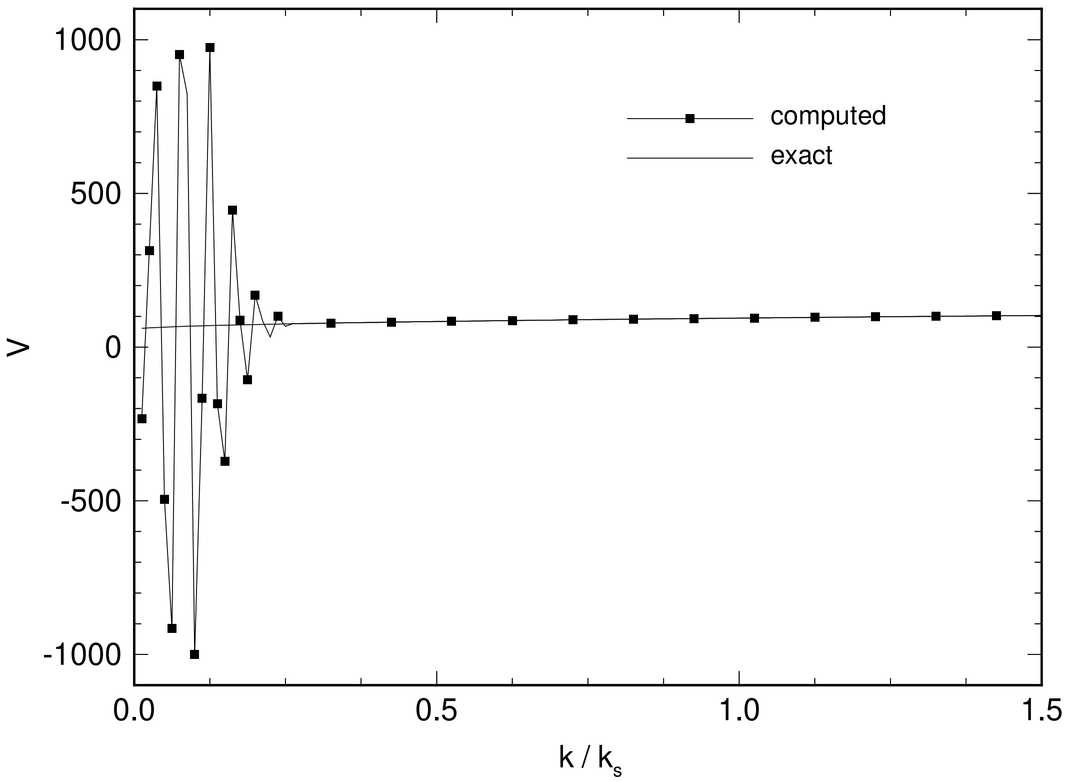
- Solve explicitly for V_i^{n+1}, $\quad i = 0, 1, 2, \ldots, i_{max}$, given an initial guess for V_i^0, and boundary conditions at $k = 0, k_{max}$

Code

- Initialize:
  - make grid: $\Delta k = \frac{k_{max}}{i_{max}}; \quad k_{max} = 2k_s; \quad k_i = i\,\Delta k$
  - set initial values of V_i: say $V_i = A\sqrt{k_i}$
- Solve PDE:
  - compute time step: $\Delta t = \mathrm{CFL}\frac{\Delta k}{a}$ (CFL is a constant, to be discussed later)
  - compute change in solution: $\Delta V_i^n = V_i^{n+1} - V_i^n, \quad \text{for all } i$
  - compute residual (L_2 error norm): $\varepsilon^n = \frac{1}{i_{max}\,\Delta t}\sqrt{\sum_i \left(\Delta V_i^n\right)^2}$
  - update solution: $V_i^{n+1} = V_i^n + \Delta V_i^n$
  - continue until ε^n is small (solution is no longer changing)
- Output the solution

Results Using Central Differences

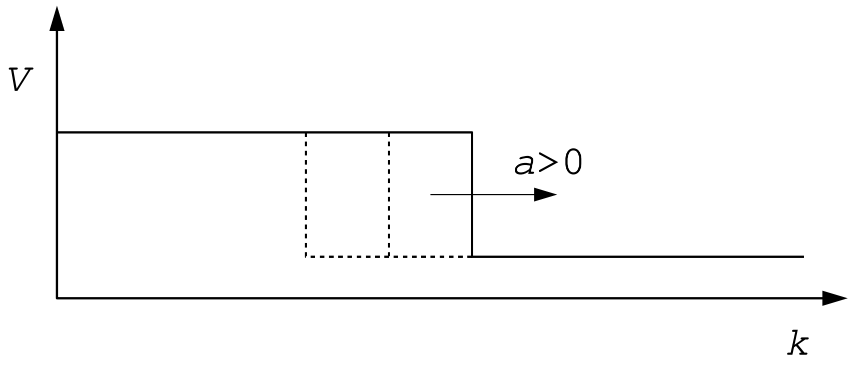
- $\omega = \theta = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Start with the exact solution and run 40 time steps:



- This is not good!

Model Equation: $V_t + aV_k = 0$

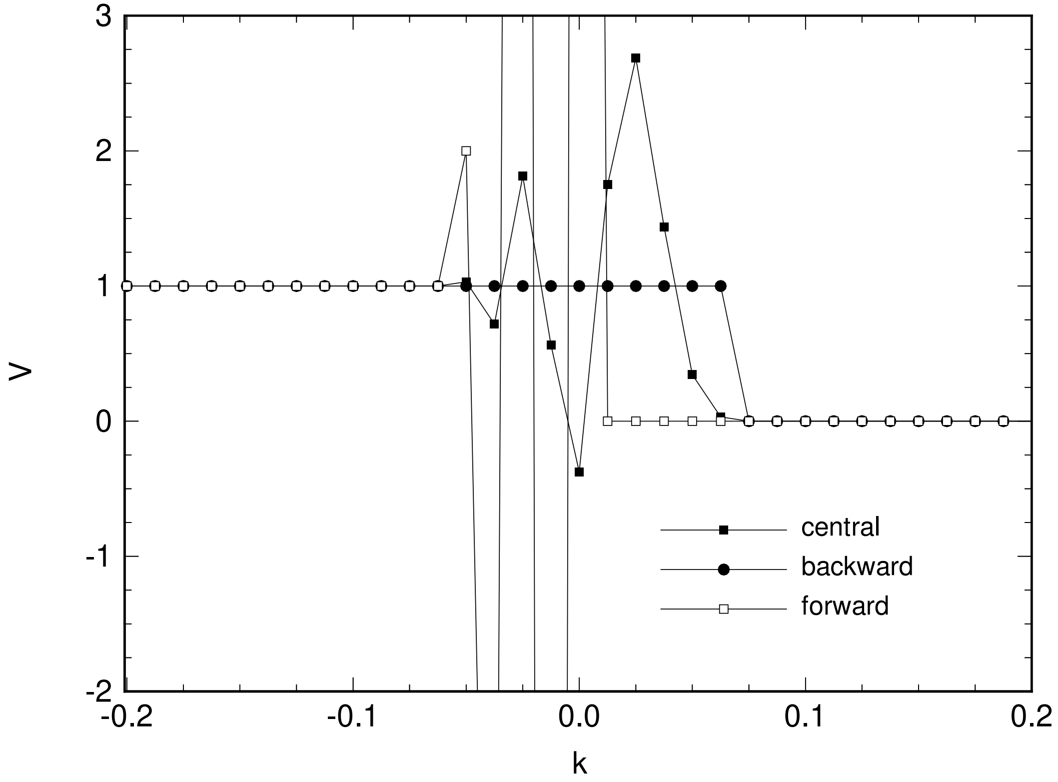
- Note that with $\sigma = 0$, there are only first derivatives in the expression for V_t
- Consider the wave equation: $V_t + aV_k = 0$



- Has an exact solution and captures the key features of the real problem.
- Try solving this problem with the central, backward, and forward difference approximations.

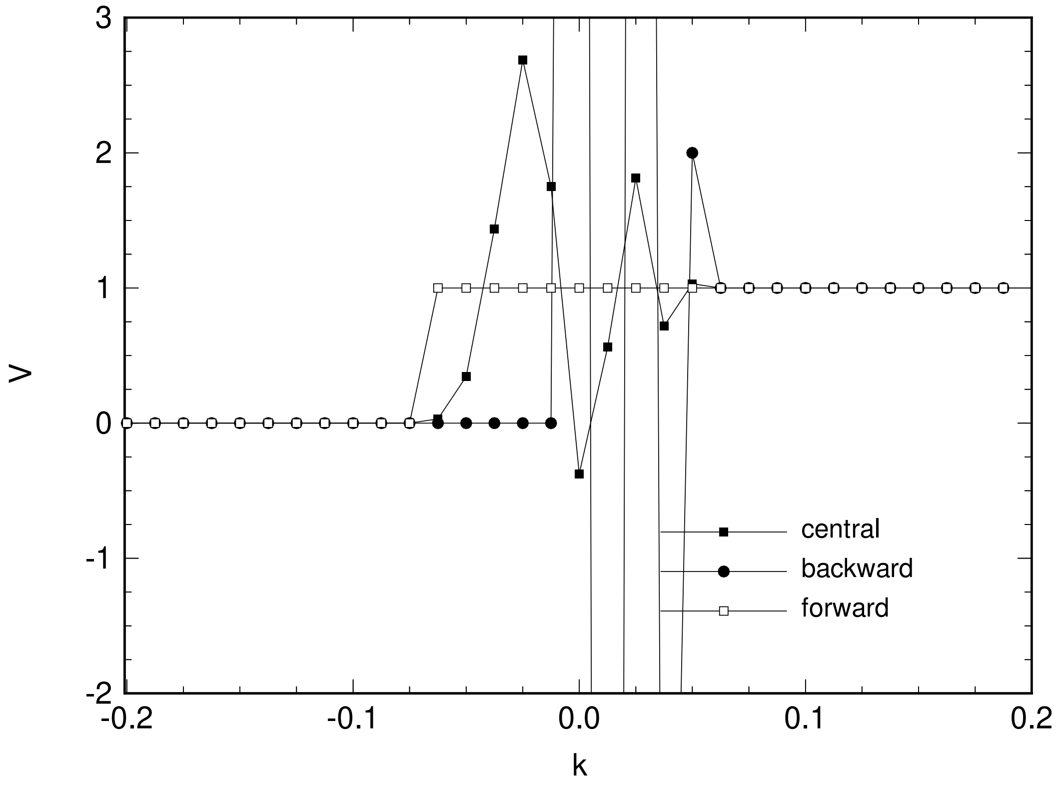
Solution of the Wave Equation ($a > 0$)

- Start with a step function and run 5 time steps:



- For $\Delta t = \frac{\Delta k}{a}$, the backward difference gives the exact solution. Other methods give spurious oscillations (for all choices of Δt).

Solution of the Wave Equation ($a < 0$)



- Now, the forward difference gives the exact solution.
- The trajectory of the solution in space-time determines the differencing scheme.

Von Neumann Stability Analysis

- It is possible to formalize the results of the example discussed above by computing the amplification factor of a finite-difference method.
- Decompose the solution, V_i^n, into the exact solution, $\bar{V}_i^n$, and the error, ϵ_i^n:

$$V_i^n = \bar{V}_i^n + \epsilon_i^n$$

  Substitute into the method, and solve for the evolution of the error.
- Consider the central difference method applied to $V_t + aV_k = 0$, $a > 0$:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = -\frac{a}{2\Delta k}\left(V_{i+1}^n - V_{i-1}^n\right)$$

  since $\bar{V}_i^n$ satisfies the equation exactly, we obtain the equation for the errors:

$$\frac{\epsilon_i^{n+1} - \epsilon_i^n}{\Delta t} = -\frac{a}{2\Delta k}\left(\epsilon_{i+1}^n - \epsilon_{i-1}^n\right)$$

- Now substitute a Fourier series representation of the error:

$$\epsilon_i^n = \sum_{j=-N}^{N} E_j^n e^{\imath\, i\phi}$$

  where $\imath = \sqrt{-1}$ and E_j^n is the amplitude of the j^{th} harmonic. $-\pi \leq \phi \leq \pi$ is the phase angle.

Von Neumann Stability Analysis

- Now, we can observe how a single harmonic is amplified by the scheme. Consider the time evolution of $E_j^n e^{\imath i\phi}$:

$$\frac{E^{n+1} - E^n}{\Delta t} e^{\imath i\phi} = -\frac{a}{2\Delta k}\left(E^n e^{\imath (i+1)\phi} - E^n e^{\imath (i-1)\phi}\right)$$
$$E^{n+1} - E^n = -\frac{a\Delta t}{2\Delta k} E^n \left(e^{\imath \phi} - e^{-\imath \phi}\right)$$

- Define the amplification factor, G, as

$$G = \frac{E^{n+1}}{E_n}$$

- The error will not grow over time if

$$|G| = \left|\frac{E^{n+1}}{E_n}\right| \leq 1 \qquad \text{for all } \phi$$

- For this case, we have

$$G = 1 - \imath \frac{a\Delta t}{\Delta k} \sin\phi$$
$$|G|^2 = 1 + \left(\frac{a\Delta t}{\Delta k}\right)^2 \sin^2\phi$$

- Therefore, for non-zero time steps the solution is always unstable. This is an unconditionally unstable scheme.

Von Neumann Stability Analysis

- For the backward difference scheme, the amplification factor is:

$$G = 1 - 2\frac{a\Delta t}{\Delta k}\sin^2\frac{\phi}{2} - \imath\,\frac{a\Delta t}{\Delta k}\sin\phi$$

  and the scheme is stable for

$$0 < \frac{a\Delta t}{\Delta k} \leq 1$$

- This result is commonly known as the Courant-Friedrichs-Lewy or CFL condition. The quantity, $\frac{a\Delta t}{\Delta k}$, is known as the CFL number or the Courant number.
- The backward difference method is said to be conditionally stable.
- Remember that Von Neumann stability analysis only works for linear problems where it is possible to superimpose the Fourier components of the error.
  - Of course, we don't care about linear problems, but the results of the stability analysis are very useful for guiding the design of the methods for nonlinear problems.
  - In practice, nonlinear problems usually have a more restrictive time step limit than that implied by the CFL condition.

Implicit Time Integration Methods

- The CFL condition can be stringent because as the mesh is refined ($\Delta k \to 0$), the time step decreases also. Therefore, many time steps may be required to obtain a converged solution.
- Consider the following differencing scheme for the linear wave equation:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = -\frac{a}{2\Delta k}\left(V_{i+1}^{n+1} - V_{i-1}^{n+1}\right)$$

- This is an implicit scheme because the solution at the new time level cannot be obtained explicitly, but rather depends on its neighbors. (We must solve a tridiagonal system of equations.)
- We get something for the effort of solving the system of equations because the amplification factor is given by:

$$G = \frac{1}{1 + \imath \, \frac{a\Delta t}{\Delta k} \sin\phi}$$

  which always satisfies the CFL condition.
- Therefore, the scheme is unconditionally stable for the linear wave equation.
  - in practice the scheme will be stable for only some range of time steps

Upwind Methods

- The foregoing analysis can be used to develop methods that sense the appropriate direction of differencing, and locally adjust the differencing scheme.
- Note that when $a > 0$ in the wave equation, we use a backward difference, and when $a < 0$ we use a forward difference.
- We can develop a scheme that automatically switches the differencing method depending on the local coefficient on V_k. For example if $a = a(k)$:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = -\frac{a_i^+}{\Delta k}\left(V_i^n - V_{i-1}^n\right) - \frac{a_i^-}{\Delta k}\left(V_{i+1}^n - V_i^n\right)$$

  where we a_i^+ and a_i^- are determined by sign of a:

$$\begin{aligned} a_i^+ &= \tfrac{1}{2}\left(a_i + |a_i|\right) \\ a_i^- &= \tfrac{1}{2}\left(a_i - |a_i|\right) \end{aligned}$$

- This is an "upwind" method since the differencing scheme uses data from the upwind direction.

Upwind Method for $V(k)$

- Time-dependent differential equation:

$$V_t + \rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} + V_k\big(\lambda k^\theta - \delta k\big) + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

- Group terms to get the coefficient on V_k:

$$V_t + \rho V = \left(\frac{1-\omega}{\omega} V_k^{1/\omega-1} + \lambda k^\theta - \delta k\right) V_k + \tfrac{1}{2}\sigma^2 k^2 V_{kk}$$

- Obtain the upwind difference method:

$$\begin{aligned}\frac{V_i^{n+1} - V_i^n}{\Delta t} + \rho V_i^n = & -\frac{a_i^+}{\Delta k}\big(V_i^n - V_{i-1}^n\big) - \frac{a_i^-}{\Delta k}\big(V_{i+1}^n - V_i^n\big) \\ & + \tfrac{1}{2}\sigma^2 k_i^2 \left(\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta k^2}\right) + O(\Delta t) + O(\Delta k)\end{aligned}$$

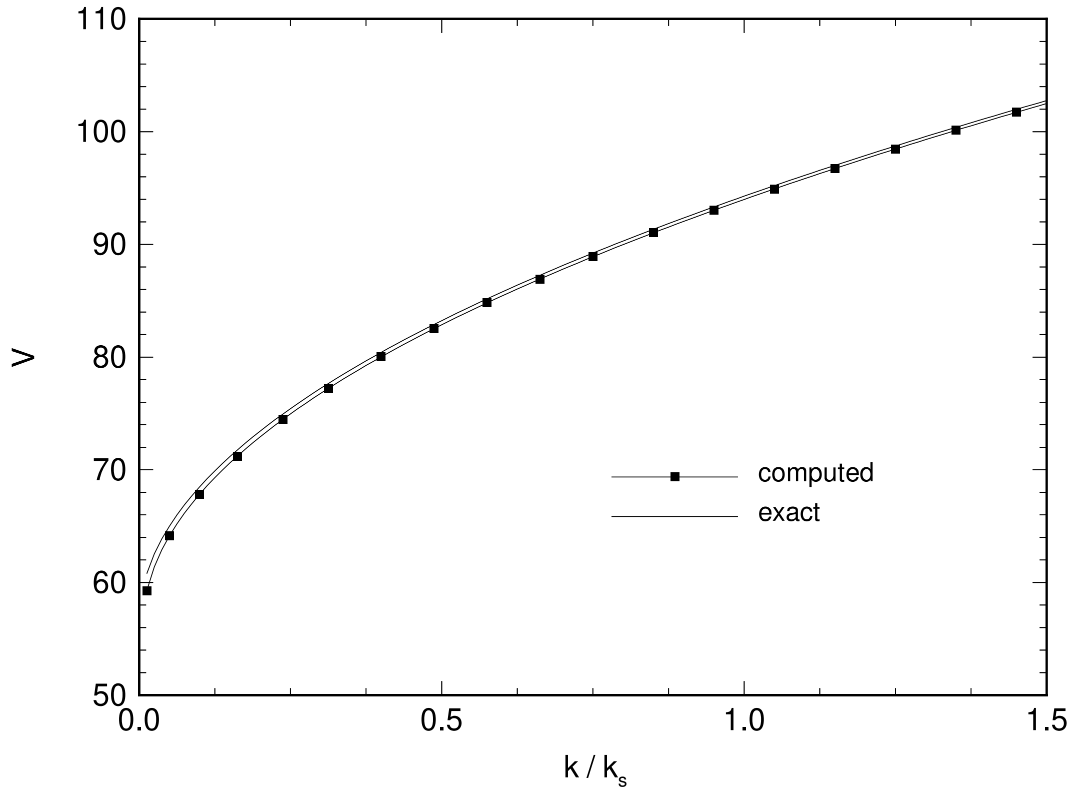
where

$$a_i = -\frac{1-\omega}{\omega}(V_k)_i^{1/\omega-1} - \lambda k_i^\theta + \delta k_i$$

- This is an extension of the upwind approach for a coefficient that depends on V_k.
- Again we can solve explicitly for V_i^{n+1} given an initial guess for V_i^0.

Results Using Upwind Method

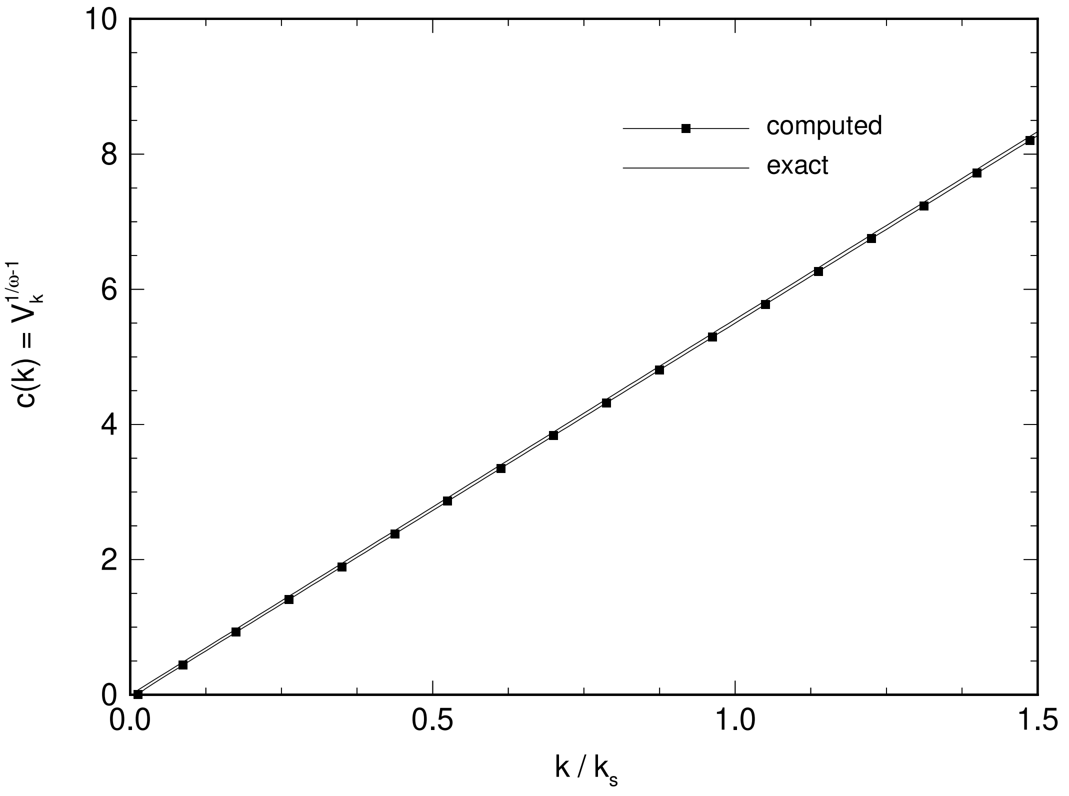
- $\omega = \theta = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Start with a guessed solution and run until error is small:



- This is good!

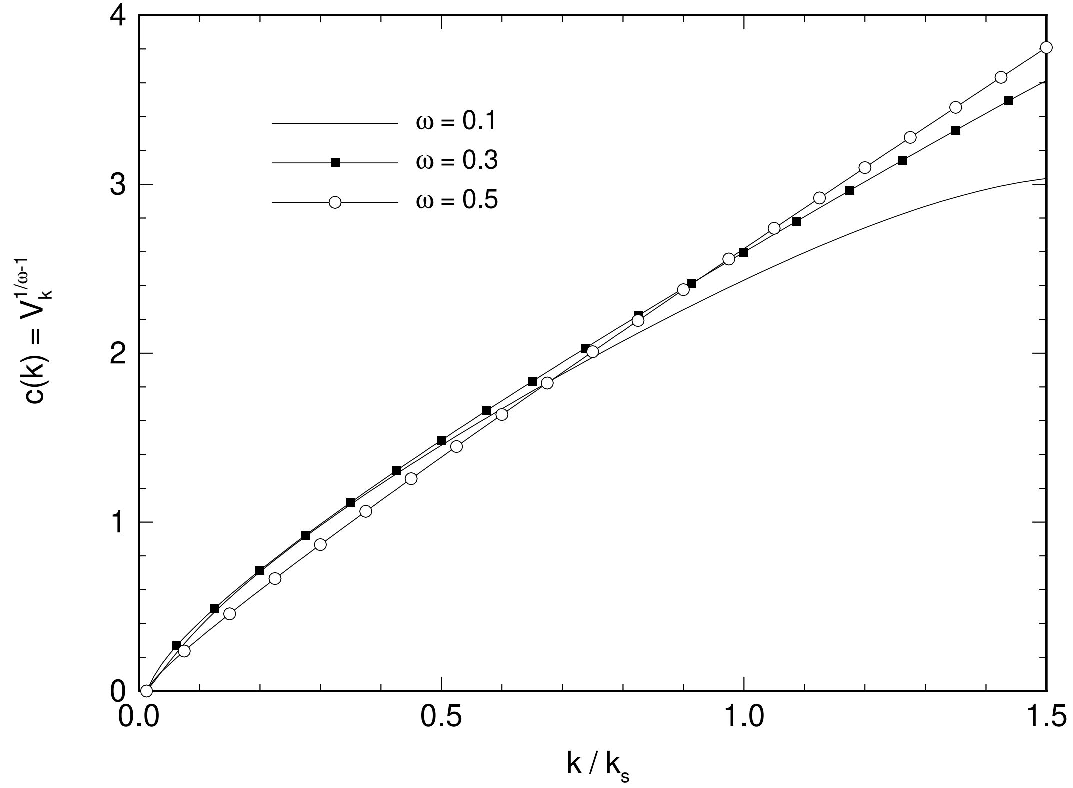
Results Using Upwind Method

- $\omega = \theta = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Consumption function:



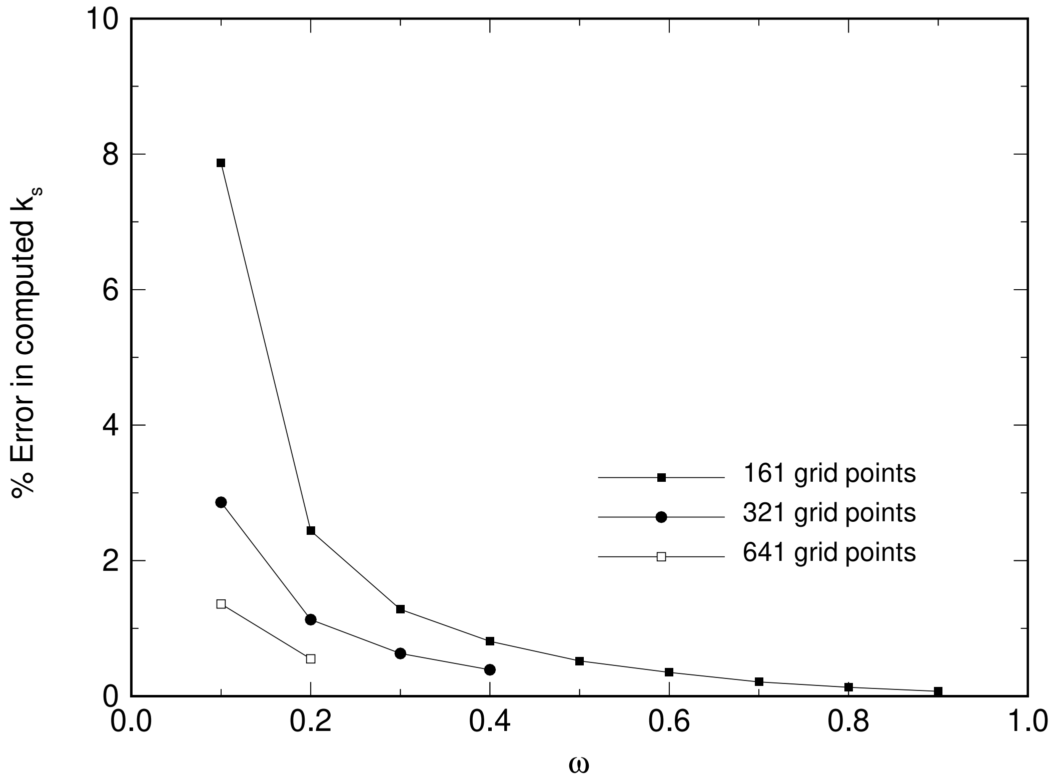
Results Using Upwind Method: ω Variation

- $\theta = 0.4$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Consumption function for various values of ω:



Results Using Upwind Method: Grid Convergence

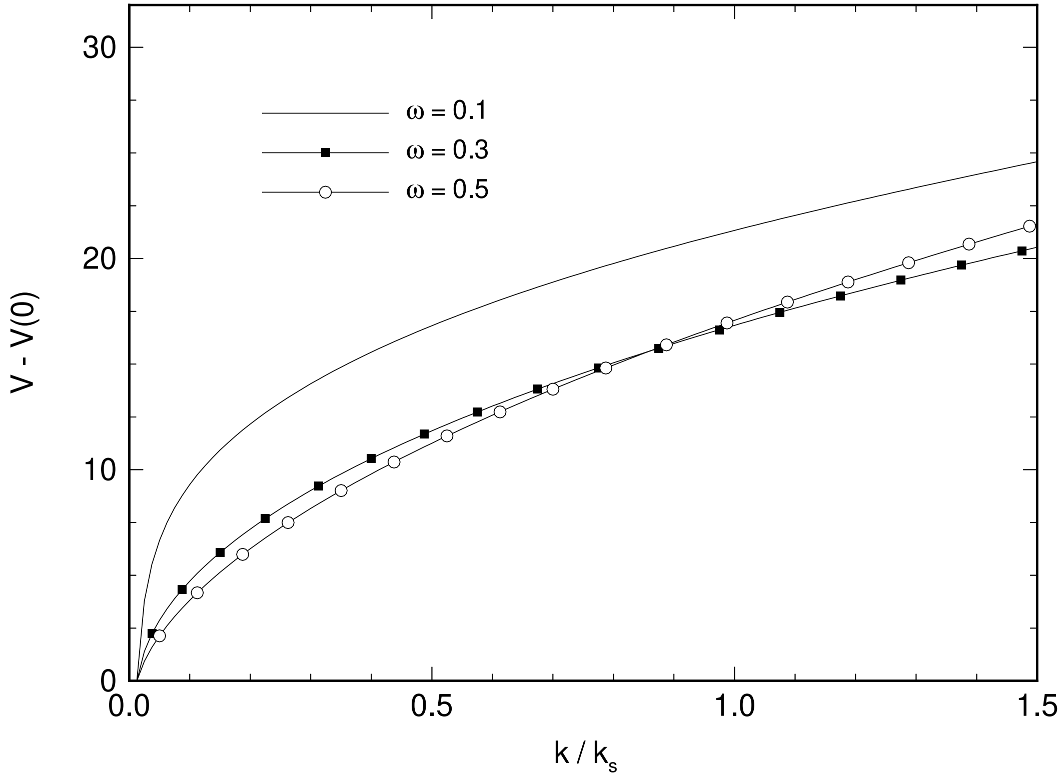
- $\theta = 0.4$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Compare computed steady-state capital stock with exact result:



- Increased grid resolution improves solution.

Results Using Upwind Method: Grid Convergence

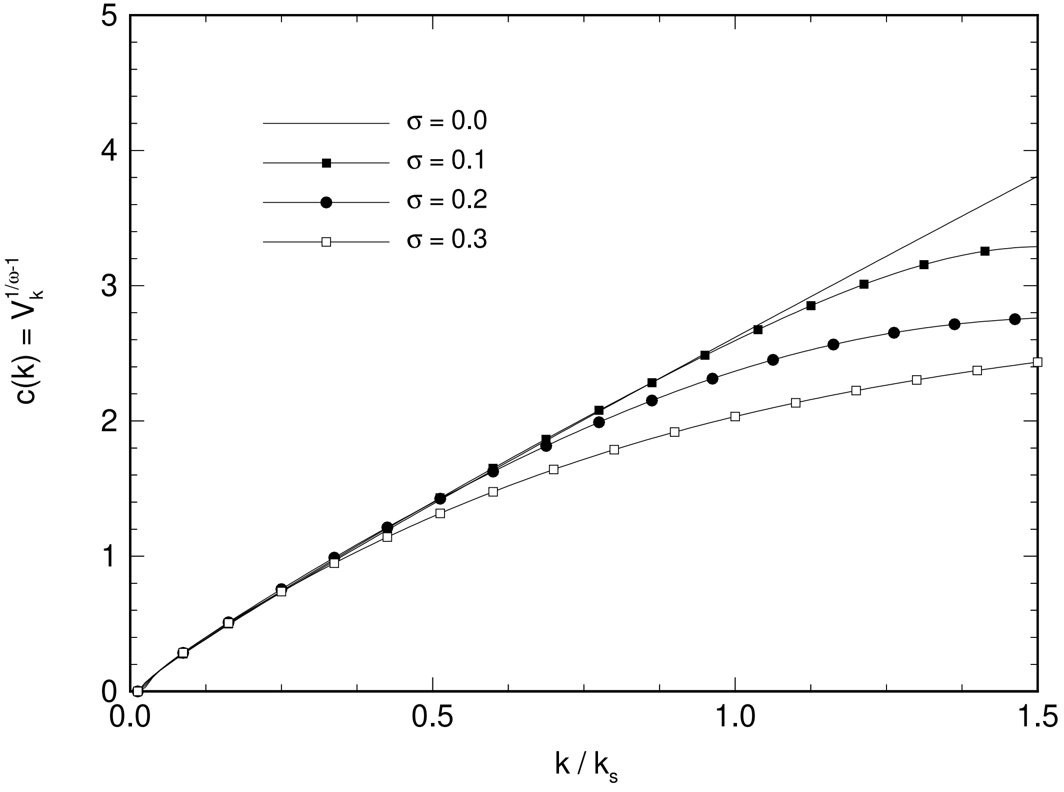
- $\theta = 0.4$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Nonlinearity of $V(k)$ increases with smaller values of ω:



- Increased error is a result of increased nonlinearity.
  - error term in first-order differencing scheme is: $\frac{1}{2}\Delta k V_{kk} = O(\Delta k)$

Results Using Upwind Method: σ Variation

- $\theta = 0.4$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$



Implicit Method for $V(k)$

- Since we are integrating the PDE from a guessed initial condition to the final answer, if we can take larger time steps, we should be able to get to the answer faster.
- We discussed an implicit method above and saw that for the wave equation, it is unconditionally stable.
    - of course, that was for a linear problem
- Let's try developing an implicit method for this problem:
    - use the same upwind differencing method for the k derivatives
    - evaluate the derivatives at the future $(n+1)$ time level

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + \rho V_i^{n+1} = -\frac{a_i^+}{\Delta k}\left(V_i^{n+1} - V_{i-1}^{n+1}\right) - \frac{a_i^-}{\Delta k}\left(V_{i+1}^{n+1} - V_i^{n+1}\right)$$
$$+ \tfrac{1}{2}\sigma^2 k_i^2 \left(\frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}}{\Delta k^2}\right) + O(\Delta t) + O(\Delta k)$$

for simplicity, use $a_i^\pm$ at the present (n) time level

Implicit Method for $V(k)$

- If we let $\Delta V_i^n = V_i^{n+1} - V_i^n$, we can write the implicit problem as:

$$\begin{aligned}\big(\alpha_i^- - \beta_i\big)\Delta V_{i+1}^n + \big(1 + \alpha_i^+ - \alpha_i^- + 2\beta_i + \Delta t\rho\big)\Delta V_i^n + \big(-\alpha_i^+ - \beta_i\big)\Delta V_{i-1}^n = \\ -\alpha_i^+\big(V_i^n - V_{i-1}^n\big) - \alpha_i^-\big(V_{i+1}^n - V_i^n\big) + \beta_i\big(V_{i+1}^n - 2V_i^n + V_{i-1}\big) - \Delta t\rho V_i^n\end{aligned}$$

  where $\alpha^\pm = \frac{\Delta t\, a^\pm}{\Delta k}$ and $\beta = \frac{\Delta t\, \sigma^2 k^2}{2\Delta k^2}$
- This is a tridiagonal system for ΔV_i^n:
  - again, the solution cannot be explicitly computed
  - tridiagonal system is easy and efficient to solve
- Symbolically, we have:

$$\begin{pmatrix} B & A & C & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_i & A_i & C_i & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B & A & C \end{pmatrix}^n \begin{pmatrix} \vdots \\ \Delta V_{i-1} \\ \Delta V_i \\ \Delta V_{i+1} \\ \vdots \end{pmatrix}^n = \begin{pmatrix} \vdots \\ RHS_i \\ \vdots \end{pmatrix}^n$$

Other Implicit Methods for $V(k)$

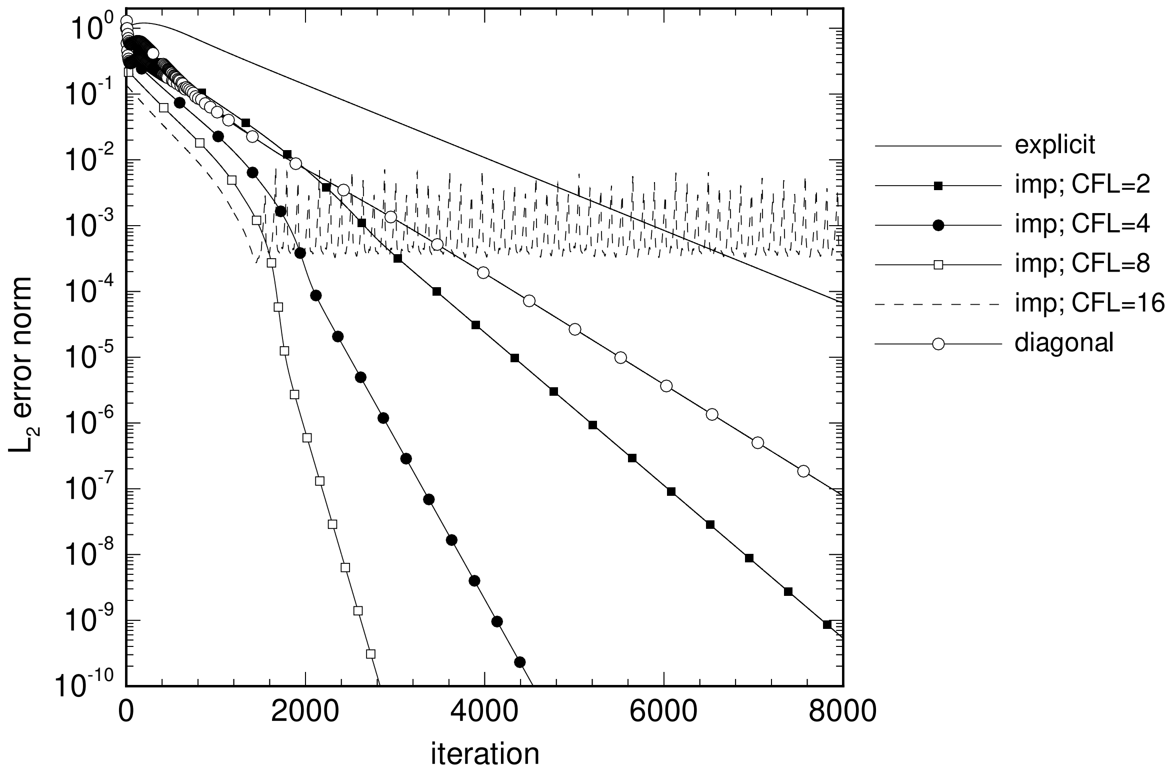
- Many other implicit methods can be derived.
  - not really needed for this one-dimensional problem, but can be useful for higher dimensions.
- Answer is independent of implicit operator.
- Note that the system is diagonally dominant for any Δt:
  - off-diagonal terms may be moved to the *RHS*
  - may be lagged in iteration number or even ignored
  - a direct result of the upwind difference method
- A “diagonal” method may be derived by ignoring off-diagonal terms:

$$
\begin{aligned}
&\big(1 + \alpha_i^+ - \alpha_i^- + 2\beta_i + \Delta t \rho\big)\Delta V_i^n = \\
&\quad - \alpha_i^+\big(V_i^n - V_{i-1}^n\big) - \alpha_i^-\big(V_{i+1}^n - V_i^n\big) + \beta_i\big(V_{i+1}^n - 2V_i^n + V_{i-1}\big) - \Delta t \rho V_i^n
\end{aligned}
$$

  - trivial to solve
  - ignores off-diagonal coupling in implicit operator
  - if $\Delta t \to \infty$, becomes a relaxation method
  - generalizes to multiple dimensions

Comparison of Convergence Rates: L_2 Error Norm

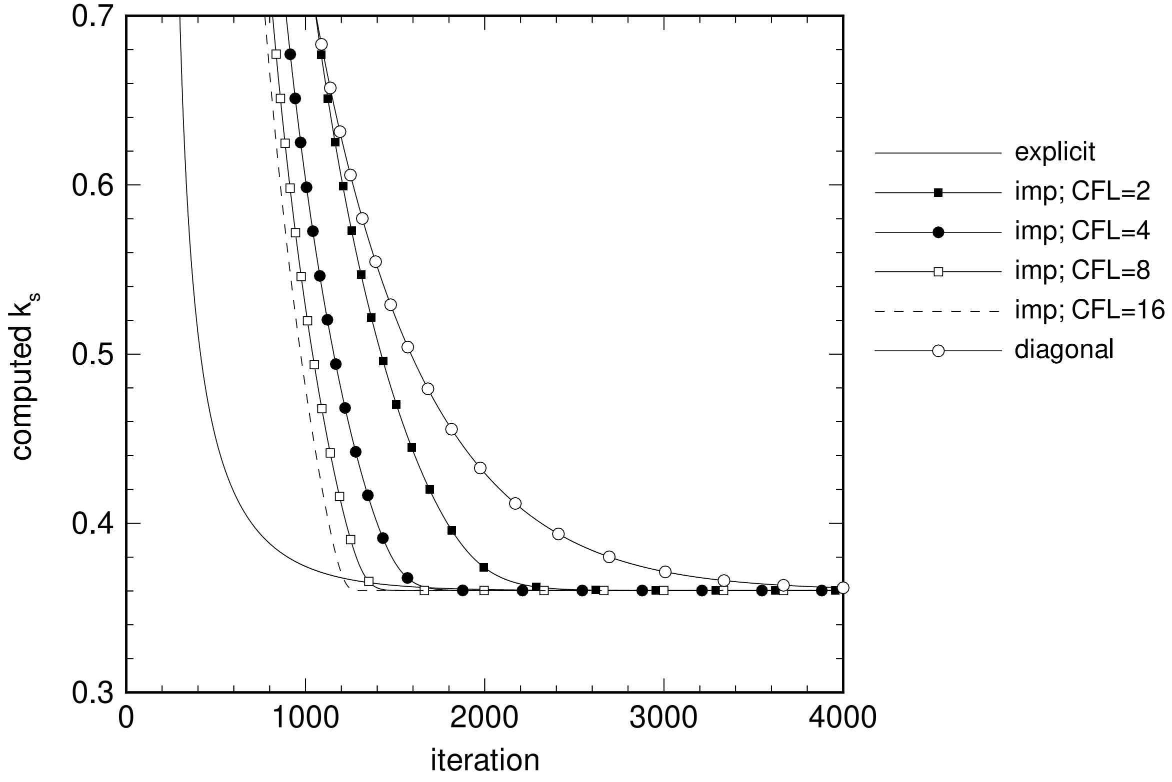
- $\theta = 0.4$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$



- The implicit method is best, but there is an optimal CFL.
- The diagonal method is slower to converge, but is stable for any time step.
- No accounting for cost has been made here.

Comparison of Convergence Rates: k_s

- $\theta = 0.4$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$



- By this measure, the diagonal method is the slowest.

Convergence Rates

- Even for the implicit method, about 2000 iterations are required to obtain a steady-state solution.
  - a very poor initial guess for V was used
  - these PDEs are different than those in physics: $V_k^{\omega/\omega-1}$
  - non-uniform grid spacing would reduce the cost significantly (cluster grid points near regions of strong nonlinearity).
  - coefficient on V_k could be linearized and included in the implicit solution
- Convergence is not all that much different for difficult nonlinear problems in computational physics.
- It is is likely that with more work better convergence rates could be obtained.
- In any case, the solutions take a few seconds on a workstation.

Upwind Method for $c(k)$

- When $\sigma = 0$, derive the time-dependent differential equation:

$$c_t = \frac{u'(c)}{u''(c)}\big(f'(k) - \rho\big) + \big(f(k) - c(k)\big)c'(k)$$

with $u(c) = c^\omega/\omega$ and $f(k) = \lambda k^\theta - \delta k$

- Obtain:

$$c_t = \big(\lambda k^\theta - \delta k - c\big)c_k + \frac{1}{\omega - 1}\big(\lambda\theta k^{\theta-1} - \delta - \rho\big)c$$

- Upwind finite-difference method:

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = -\frac{a_i^+}{\Delta k}\big(c_i^n - c_{i-1}^n\big) - \frac{a_i^-}{\Delta k}\big(c_{i+1}^n - c_i^n\big) + S_i^n$$

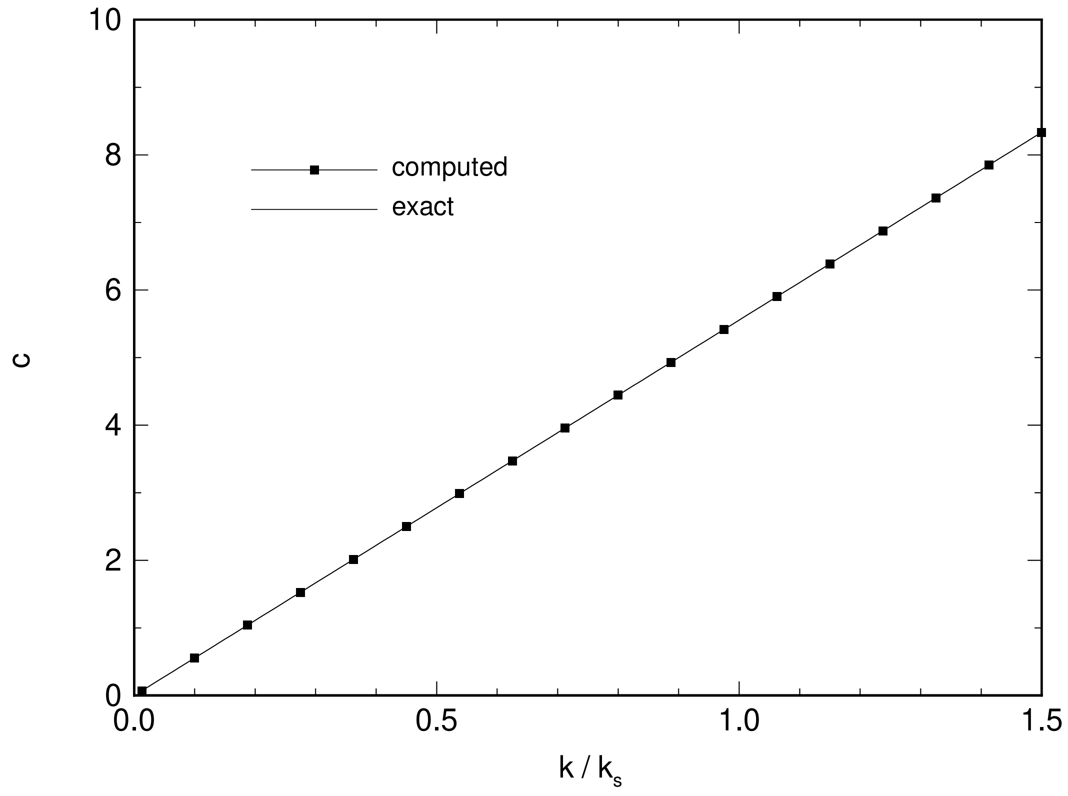
where

$$a_i = -\lambda k_i^\theta + \delta k_i + c_i$$

$$S_i = \frac{1}{\omega - 1}\big(\lambda\theta k_i^{\theta-1} - \delta - \rho\big)c_i$$

Results Using Upwind Method

- $\omega = \theta = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0$



- This is good!

Implicit Method for $c(k)$

- Again evaluate all quantities at future time level:

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = -\frac{a_i^+}{\Delta k}\left(c_i^{n+1} - c_{i-1}^{n+1}\right) - \frac{a_i^-}{\Delta k}\left(c_{i+1}^{n+1} - c_i^{n+1}\right) + S_i^{n+1}$$

- Linearize the source term:

$$S_i^{n+1} = S_i^n + \left.\frac{\partial S}{\partial c}\right|_i^n \left(c_i^{n+1} - c_i^n\right) + O(\Delta t)$$

- Obtain:

$$\alpha_i^- \Delta c_{i+1}^n + \left(1 + \alpha_i^+ - \alpha_i^- - \Delta t \frac{\partial S}{\partial c}\right) \Delta c_i^n - \alpha_i^+ \Delta c_{i-1}^n = \\ - \alpha_i^+ \left(c_i^n - c_{i-1}^n\right) - \alpha_i^- \left(c_{i+1}^n - c_i^n\right) + \Delta t S_i^n$$

- The convergence rate is poor: $\mathrm{CFL}_{max} \simeq 2$

Model Equation for $c(k)$

- Consider the case where c_k is small. Then:

$$c_t = \frac{1}{\omega - 1}\left(\lambda\theta k^{\theta-1} - \delta - \rho\right)c = sc = S$$

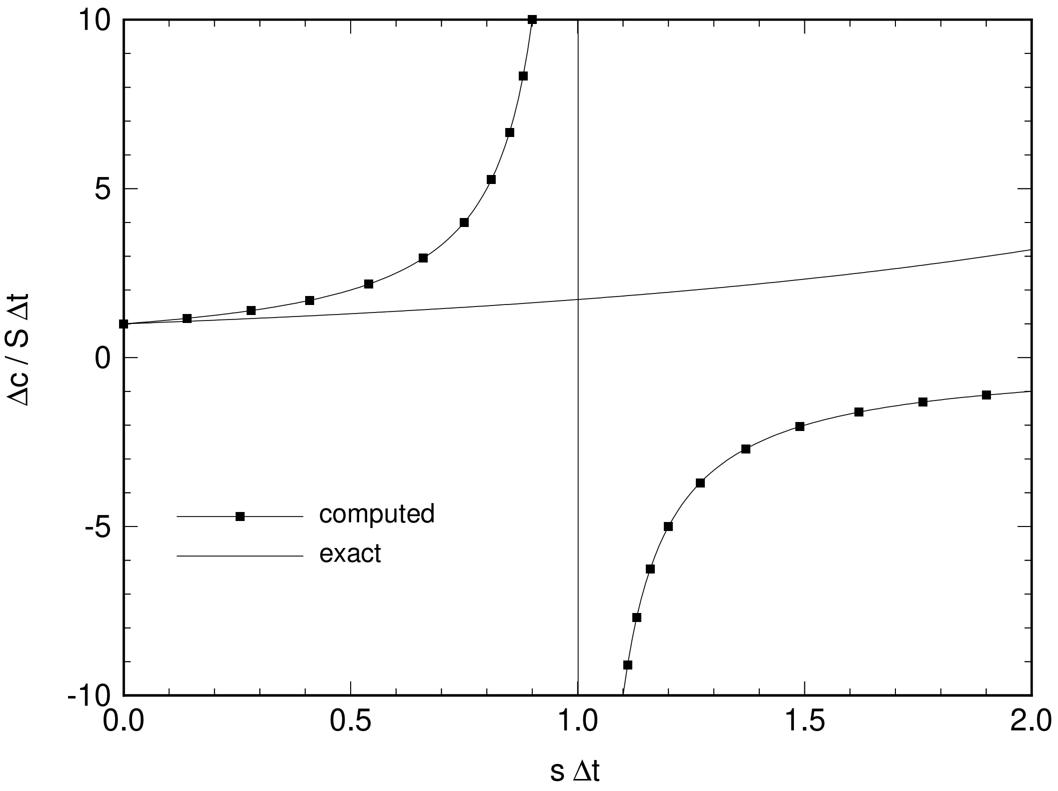
- An implicit representation:

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = sc^{n+1} \simeq S^n + s\left(c_i^{n+1} - c_i^n\right)$$
$$c^{n+1} = c^n + \frac{\Delta t S^n}{1 - s\Delta t}$$

- If $s > 0$ and Δt is large, can have problems.

Model Equation for $c(k)$

- For constant s, we can compare to exact solution:

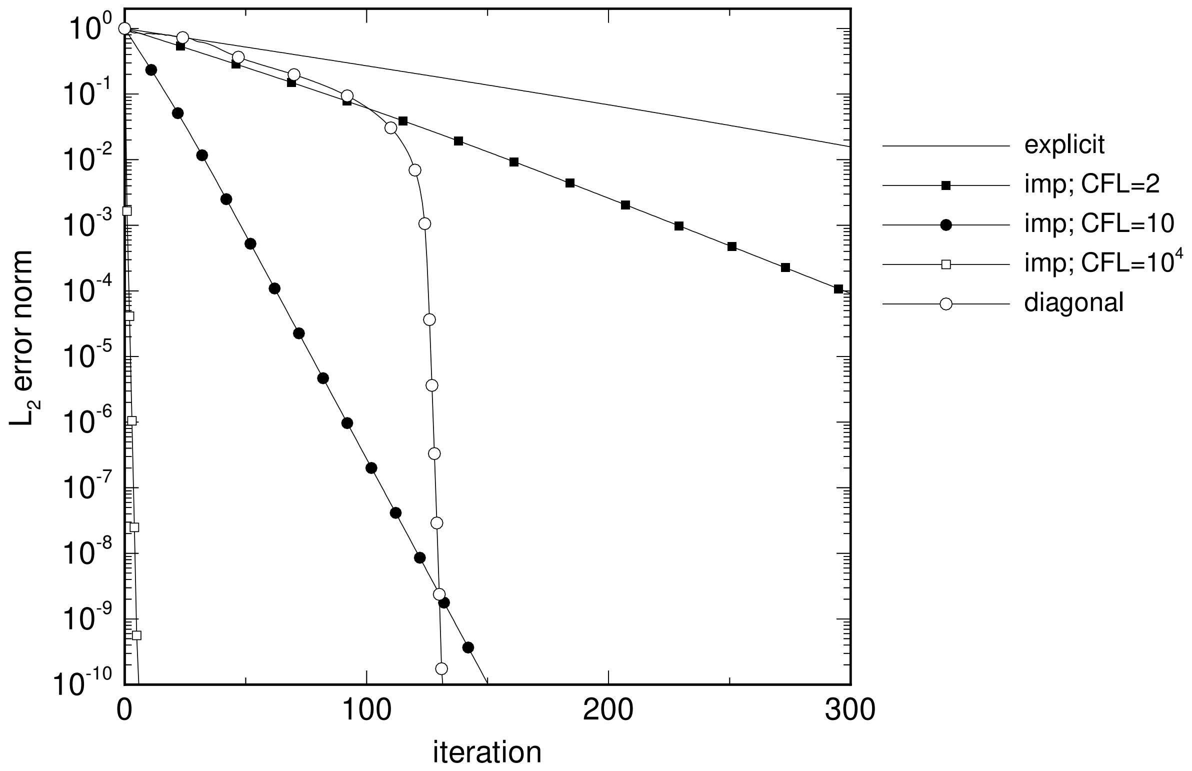


- Clearly, this approach has problems for $s\Delta t$ large.

Comparison of Convergence Rates: L_2 Error Norm

- $\theta = 0.4$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0$
- Let

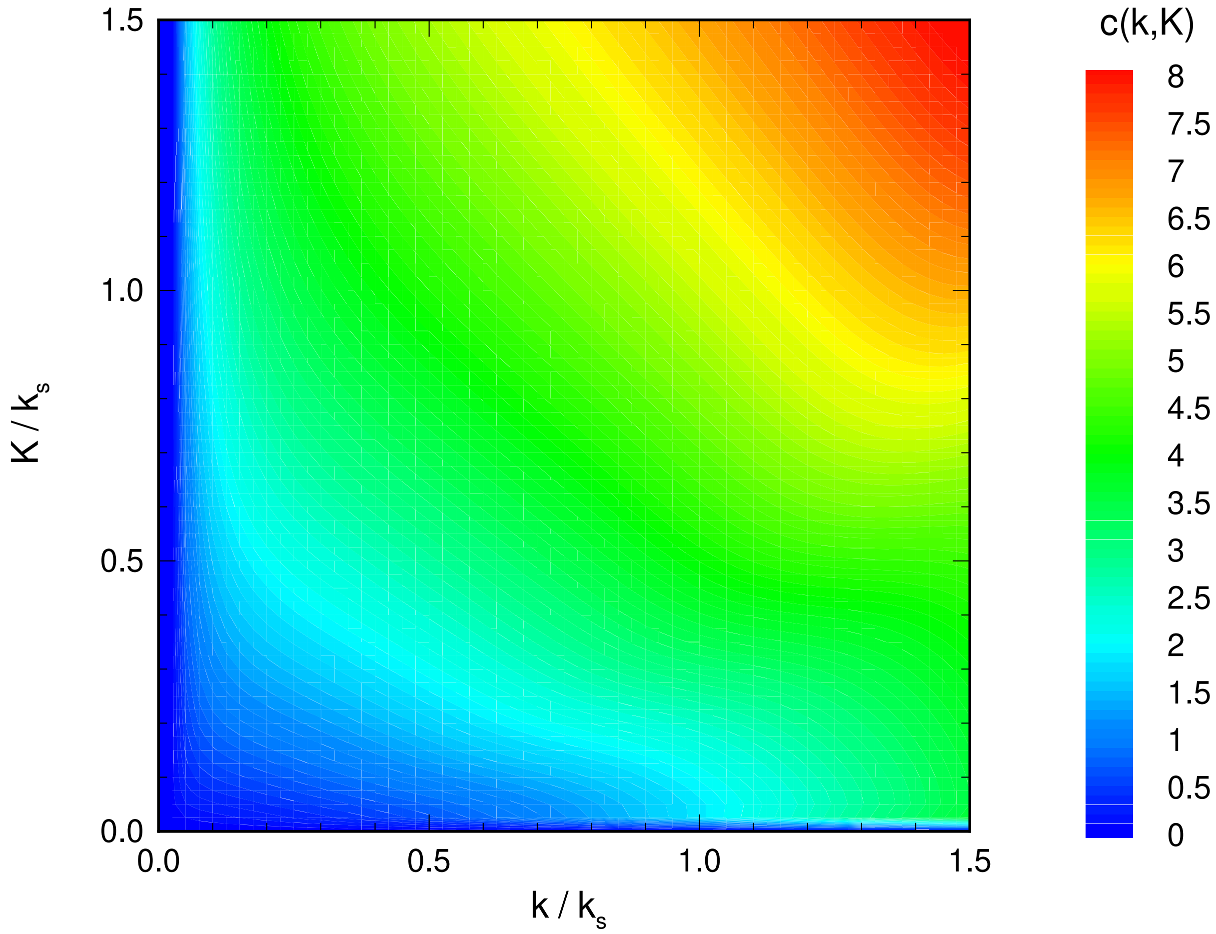
$$\frac{\partial S}{\partial c} \sim \frac{1}{\omega - 1} \lambda \theta k^{\theta - 1} < 0$$



- For this case, the implicit methods are much better.

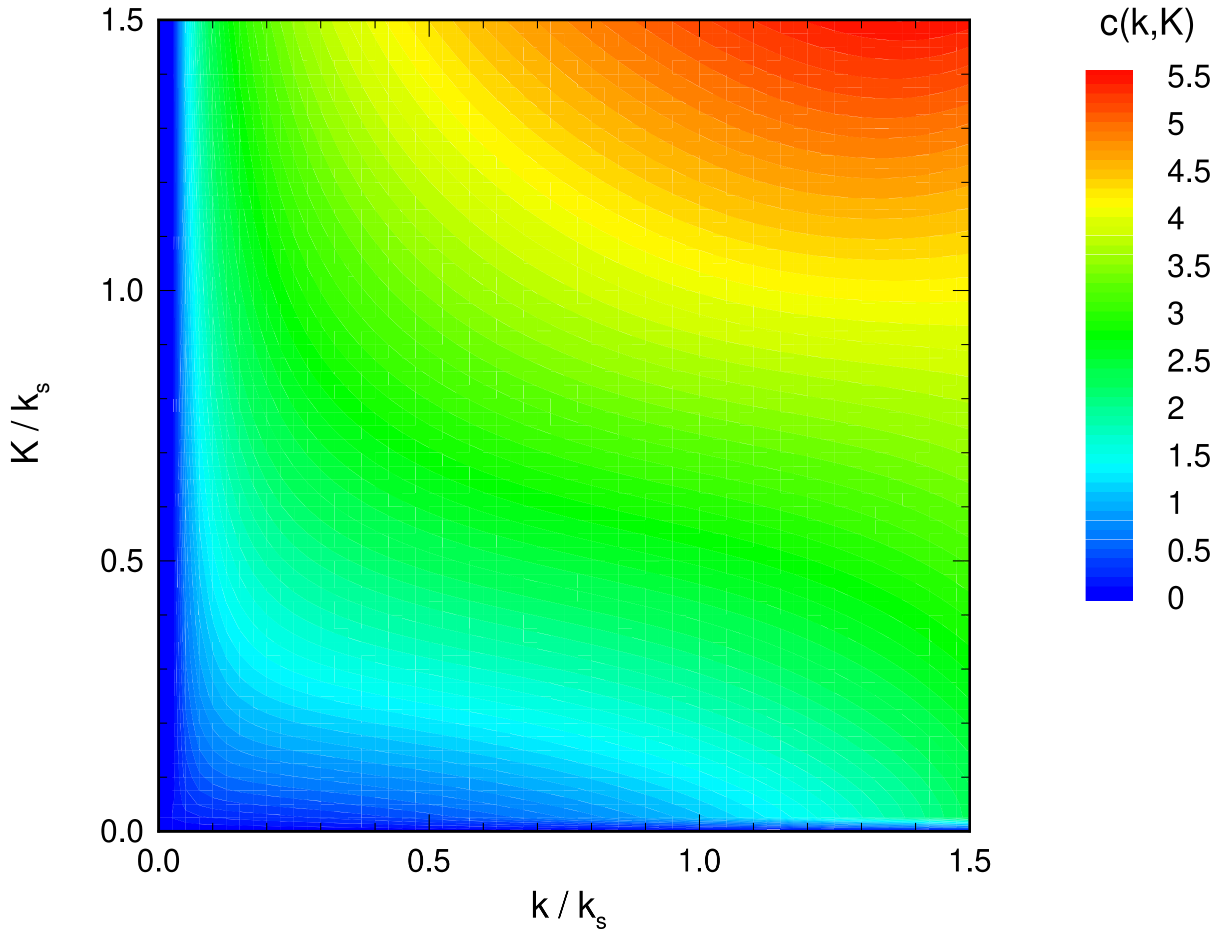
Results Using 2-D Upwind Method: $V(k, K)$

- $\theta = 0.5$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0$, $\tau = 0.0$



Results Using 2-D Upwind Method: $V(k, K)$

- $\theta = 0.5$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0$, $\tau = 0.4$



Stochastic Dynamic Programming (2-D)

- Agent's Problem:

$$\max_c E \int_0^\infty e^{-\rho t}\, u(c)\, dt$$

$$\text{s.t.} \quad dk = \big((1-\tau)r(K)k + w(K) - c + T(K)\big)dt + \sigma(k,K)dz$$

$$dK = \big(F(K) - C(K) - G(K)\big)dt + \sigma(K,K)dz$$

where:

$$\begin{aligned} c &= \text{individual consumption} \\ k &= \text{individual capital stock} \\ C(K) &= \text{aggregate consumption} \\ K &= \text{aggregate capital stock} \\ F(K) &= \text{aggregate output} \\ G(K) &= \text{government spending} \\ T(K) &= \text{transfers} \end{aligned}$$

let:

$$u(c) = c^\omega/\omega, \quad F(k) = \lambda k^\theta - \delta k, \quad r = F_K, \quad w = F - F_K K, \quad G = \tau F_K K$$

Stochastic Dynamic Programming (2-D)

- The general dynamic programming equation for the value function, $V(k, K)$, is:

$$
\begin{aligned}
V_t + \rho V = \max_c \Big\{ u(c) &+ V_k\big((1-\tau)(\theta\lambda K^{\theta-1} - \delta)k + \lambda K^\theta(1-\theta) - c\big) \\
&+ V_K\big(\lambda K^\theta - C(K) - \tau(\theta\lambda K^\theta - \delta K)\big) \\
&+ \tfrac{1}{2}\big(\sigma(k,K)^2 V_{kk} + 2\sigma(k,K)\sigma(K,K)V_{kK} + \sigma(K,K)^2 V_{KK}\big)\Big\}
\end{aligned}
$$

- The first-order condition, $u'(c) = V_k(k, K)$, gives the differential equation:

$$
\begin{aligned}
V_t + \rho V = \frac{1-\omega}{\omega} V_k^{\omega/\omega-1} &+ V_k\big((1-\tau)(\theta\lambda K^{\theta-1} - \delta)k + \lambda K^\theta(1-\theta)\big) \\
&+ V_K\big(\lambda K^\theta - \delta K - V_k(K,K)^{1/\omega-1} - \tau(\theta\lambda K^\theta - \delta K)\big) \\
&+ \tfrac{1}{2}\big(\sigma(k,K)^2 V_{kk} + 2\sigma(k,K)\sigma(K,K)V_{kK} + \sigma(K,K)^2 V_{KK}\big)
\end{aligned}
$$

Finite-Difference Solution of $V(k, K)$

- Now let:

$$V(t,k,K) = V(t_n, k_i, K_j) = V(n\,\Delta t, i\,\Delta k, j\,\Delta K) = V^n_{i,j}$$

- Derive the upwind difference method:

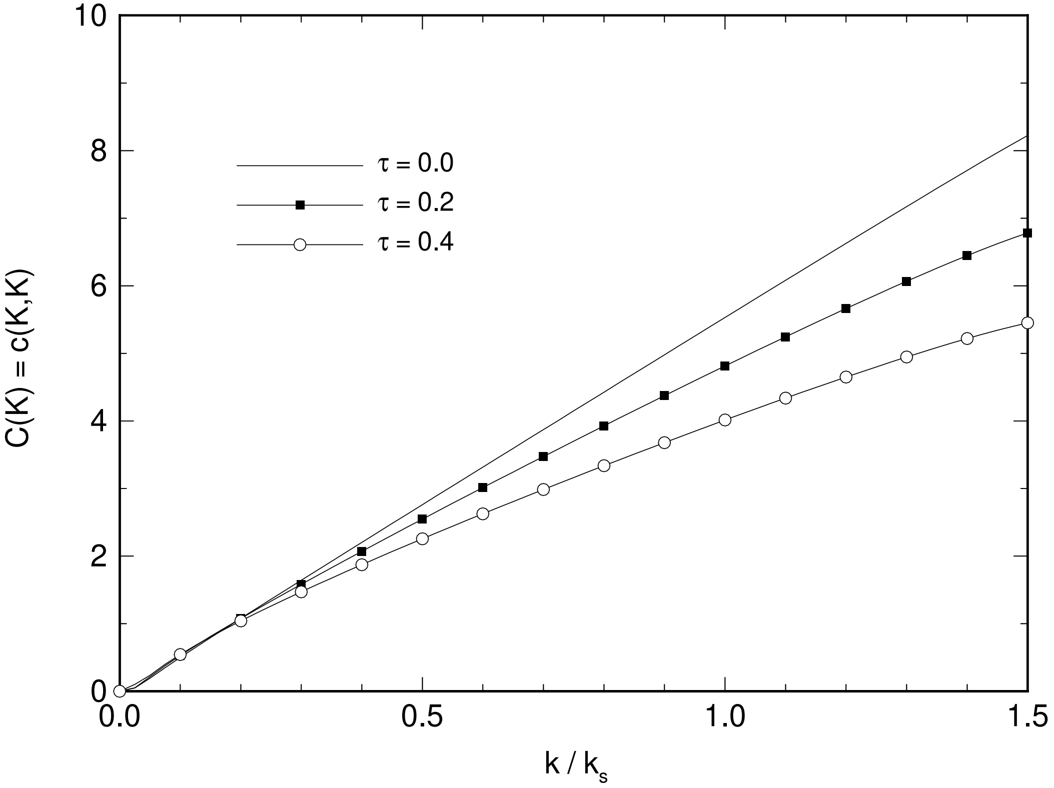
$$\begin{aligned}\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta t} + \rho V^n_{i,j} = &- \frac{a^+_{i,j}}{\Delta k}\left(V^n_{i,j} - V^n_{i-1,j}\right) - \frac{a^-_{i,j}}{\Delta k}\left(V^n_{i+1,j} - V^n_{i,j}\right) \\ &- \frac{b^+_{i,j}}{\Delta K}\left(V^n_{i,j} - V^n_{i,j-1}\right) - \frac{b^-_{i,j}}{\Delta K}\left(V^n_{i,j+1} - V^n_{i,j}\right)\end{aligned}$$

where

$$\begin{aligned} a_{i,j} &= -\frac{1-\omega}{\omega}(V_k)^{1/\omega-1}_{i,j} - (1-\tau)(\theta\lambda K^{\theta-1}_{i,j} - \delta)k_{i,j} - \lambda K^\theta_{i,j}(1-\theta) \\ b_{i,j} &= -\lambda K^\theta + \delta K + V_k(K,K)^{1/\omega-1} + \tau(\theta\lambda K^\theta - \delta K)\end{aligned}$$

Results Using 2-D Upwind Method: τ Variation

- $\theta = 0.5$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- Aggregate consumption function for various values of τ:



Results Using 2-D Upwind Method: τ Variation

- $\theta = 0.5$, $\omega = 0.5$, $\rho = 0.05$, $\lambda = 1.0$, $\delta = 0.025$, $\sigma = 0.0$
- For $\tau = 0.0$, compare to exact solution: