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APPENDICES
Sticky Price Models of the Business Cycle: Can the Contract Multiplier Solve the Persistence Problem?

V. V. Chari*

University of Minnesota
and Federal Reserve Bank of Minneapolis

Patrick J. Kehoe*

University of Pennsylvania,
Federal Reserve Bank of Minneapolis,
and National Bureau of Economic Research

Ellen R. McGrattan*

Federal Reserve Bank of Minneapolis

*The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

Appendix A. Sensitivity Analysis

In this appendix we compute the contract multiplier for a wide range of parameters in the benchmark economy, the economy with convex demand, and the economy with specific factors. We use the parameters in Table 1 of the text as the baseline and vary each parameter of each model. For all of the experiments described below (with the exception of varying b) we adjust the parameter b so that the standard deviation of investment is 3.23 times that of output – as it is in the postwar U.S. data.

In Figure A1, we display the contract multipliers for the benchmark economy. Panel 1 shows the multiplier as we vary the interest elasticity of money demand (η). Notice that the multiplier depends nonlinearly on the elasticity. The value changes little until we reach a very low elasticity and then it increases significantly. In panel 2, we display the multiplier for different capital share values. It reaches a peak at $\alpha = 0.2$ but the multiplier there is not much different than in the baseline case with $\alpha = 0.33$. In panel 3, we vary the demand elasticity. As we noted in the text, capital acts much like a specific factor. Therefore, when capital is included, the benchmark economy behaves similarly to the specific factors economy. In both, one needs a very large demand elasticity to get a large multiplier. In this case, with a demand elasticity of 300 – 30 times the baseline setting – we find a multiplier close to 2.5. In panel 4, we vary the risk aversion parameter. When $\sigma = 2.2$, no adjustment costs are needed to reduce the volatility of investment. Therefore, the multiplier is very small for higher values of σ. When σ is close to 0, the multiplier rises to about 1.3.

For panels 5-8 of Figure A1, we compute the contract multiplier as we vary the weight on leisure in utility ψ, the adjustment cost parameter b, the serial correlation of money growth ρ, and the weight on consumption when we aggregate consumption and money in utility ω. The multiplier increases to 1.27 as we drive ψ to 99. For values of 25 and above the relation between the multiplier and ψ is very flat. This is shown in panel 5. In panel 6, we show the results as we vary the adjustment cost parameter. Our baseline case gives a standard deviation of investment relative to output of 3.23 as in the data. Increasing the costs smooths out investment and increases the multiplier. If we increase the costs so as to get the standard deviation of investment relative to output down below 2.5, the multiplier increases from 1.06 (the baseline) to 1.2. Notice also that the multiplier quickly falls off as we reduce the adjustment costs and allow for greater volatility of investment. The multiplier is below 1 even for a standard deviation of 3.5. In panels 7 and 8 we vary ρ and ω. As the scales show, there is very little change in the multiplier.

Finally, in panels 9 and 10 of Figure A1, we display the multiplier for the discount factor β and the depreciation rate δ both reported as annualized values. To increase the multiplier, we need to reduce the discount factor and the depreciation rate. If the annual discount factor is 0.9, then the multiplier in the benchmark economy is 1.25. We get a similar result with an annual depreciation rate of 0.02.

In Figure A2, panels 1-11, we report results for the economy with convex demand. Again, the baseline parameters are given in Table 1 of the text. The only difference between the baseline of this case and that for Figure A1 is our setting of the demand curvature parameter χ. In the benchmark economy, $\chi = 1 + \varepsilon = 11$ and in the economy with convex demand $\chi = -289$.

Each panel of Figure A2 shows how the multiplier changes as we vary each of the parameters – holding fixed the volatility of investment relative to output. Qualitatively, the results for variations in η, σ, ψ, b, ω, β, and δ are very similar to the results with the benchmark economy. (See panels 1, 4, 5, 6, 8, 9, and 10 in Figures A1 and A2.) Quantitatively, we find that the maximum multipliers in the convex-demand economy are about double what they are in the benchmark economy (with the baseline multiplier equal to 1.06 for the benchmark and 1.87 for the convex demand economy).

For panel 2 of Figure A2, we compute the multiplier as we vary the capital share. Here, the capital share plays an important role. As we drive capital out of the model the multiplier increases to 3.05 monotonically. This is one of the intertemporal links that we stressed in the text. For panel 3 of Figure A2, we display the multiplier as we vary the demand elasticity. Notice that in this case the multiplier does not vary monotonically as we vary the elasticity. It is increasing in the demand elasticity at values of ε above 20. However, higher values can be obtained by lowering the demand elasticity and, therefore, increasing the markup. For example, when the markup is as high as 1.6, the multiplier is a little over 6. As the figure shows, however, it is very nonlinear. Very high markups are needed to get the multiplier much above the baseline value.

In panel 8 of Figure A2, we show how the multiplier changes as we vary the serial correlation of money growth. Recall that in the benchmark economy, the multiplier barely changed. In the case of convex demand, there is a little more variation in the contract multiplier. At its peak the multiplier is about 2.15 – somewhat above the baseline case of 1.87.

Finally for panel 11 of Figure A2, we compute the contract multiplier as we vary the demand curvature χ. The baseline case has $\chi = -289$. We vary χ between 11 (which is the value of χ in our benchmark economy) and -46,240 which is what we needed to get a multiplier above 20. It is important to note that the adjustment costs play an important role for these results. For example, suppose that we took the case with $\chi = -46,240$ and the remaining baseline parameters for the convex demand economy. In this case, $\sigma = 1$. Now assume that $\sigma = 2.5$. If the adjustment cost parameter b is set so that investment is as volatile in the model relative to output as it is in the data, we would set $b = 0$. With these parameter settings (i.e., $\chi = -46,240$, $\sigma = 2.5$, $b = 0$), the multiplier is equal to 0.08 and the standard deviation of investment to output is equal to 3.23 as it is in the data.

The last set of experiments that we run are for the economy with specific factors. In Figure A3, panels 1-12, we display the contract multiplier as we vary all of the parameters one at a time. Here again, with the exception of panel 6, we set b so that the standard deviation of investment is 3.23 times that of output. For this economy, we again find similar qualitative results to the benchmark economy when we vary η, σ, ψ, b, ω, β, and δ. (See panels 1, 4, 5, 6, 8, 9, and 10 in Figures A1 and A3.) In most of these cases the maximum multiplier is on the order of 0.2 or 0.3 higher in the specific factors economy than in the benchmark. The main difference between these two economies is evident in the results for the factor shares and the demand elasticity. As we lower the capital and labor shares or increase the decreasing returns parameter, we increase the multiplier. With only labor, the multiplier is 1.54. An increase in the demand elasticity leads to a monotonic increase in the multiplier. However, at an elasticity of 300 – 30 times the baseline case – we find a multiplier of close to 4.5 which is significantly higher than the value of 2.5 found in the benchmark economy.

Appendix B. Derivation of Equilibrium in Economy with Zero Income Effects

In this appendix, we show for the economy with zero income effects and $N = 2$ that the solution for output, capital, and prices (in logs) is well approximated by:

$$
\begin{aligned}
&(1) \qquad & y_t &= k_t + B\mu_t \\
&(2) \qquad & k_{t+1} &= k_t + D\mu_t \\
&(3) \qquad & x_t &= x_{t-1}
\end{aligned}
$$

as we drive κ, ξ, and ν to zero. Recall that preferences in this economy are given by

$$(4) \qquad U(C,L,M) = \frac{1}{1-\sigma}\left(\frac{C^{1-\kappa}}{1-\kappa} - \omega_1 \frac{L^{1+\xi}}{1+\xi} + \omega_2 \frac{M^{1-\nu}}{1-\nu}\right)^{1-\sigma}.$$

To prove our claim, we will linearize the first-order conditions for the economy around steady state values, substitute in our guesses in equations (1)-(3), and show that as κ, ξ, and ν are driven to zero, so are the residuals of the first-order conditions.

The first-order conditions for the economy are the pricing equation for the intermediate goods producers, the money demand equation, the Euler equation for capital, and the resource constraint. Before linearizing, these equations look as follows:

$$
\begin{aligned}
(5) \qquad & P(s^{t-1})\theta Q(s^t|s^{t-1})\{\bar{P}(s^t)^{\frac{1}{1-\theta}}Y(s^t) + Q(s^{t+1}|s^t)\bar{P}(s^{t+1})^{\frac{1}{1-\theta}}Y(s^{t+1})\} \\
& \quad = Q(s^t|s^{t-1})\{\bar{P}(s^t)^{\frac{2-\theta}{1-\theta}}Y(s^t)v(s^t) + Q(s^{t+1}|s^t)\bar{P}(s^{t+1})^{\frac{2-\theta}{1-\theta}}Y(s^{t+1})v(s^{t+1})\}
\end{aligned}
$$

$$(6) \qquad U_m(s^t) - U_c(s^t) + \beta E\left\{U_c(s^{t+1})\frac{\bar{P}(s^t)}{\bar{P}(s^{t+1})} \mid \mu(s^t), \mu(s^{t-1}), \ldots\right\} = 0$$

$$(7) \qquad U_c(s^t) = E\left\{U_c(s^{t+1})\left(-\frac{\alpha}{1-\alpha}\frac{L(s^{t+1})}{K(s^t)}\frac{U_l(s^{t+1})}{U_c(s^{t+1})} + 1 - \delta\right) \mid \mu(s^t), \mu(s^{t-1}), \ldots\right\}$$

$$
\begin{aligned}
(8) \qquad & Y(s^t) = C(s^t) + K(s^t) - (1-\delta)K(s^{t-1}) \\
& \quad = AK(s^{t-1})^{\alpha}L(s^t)^{1-\alpha}\left[\frac{1}{2}\sum_{i=1}^{2}P(s^{t-i})^{\frac{\theta}{1-\theta}}\right]^{\frac{1}{\theta}}\left[\frac{1}{2}\sum_{i=1}^{2}P(s^{t-i})^{\frac{1}{1-\theta}}\right]^{-1},
\end{aligned}
$$

where the aggregate price index, unit costs, and the bond price are given by the following expressions

$$(9) \qquad \bar{P}(s^t) = \left[\frac{1}{2}P(s^{t-1})^{\frac{\theta}{\theta-1}} + \frac{1}{2}P(s^{t-2})^{\frac{\theta}{\theta-1}}\right]^{\frac{\theta-1}{\theta}}$$

$$(10) \qquad v(s^t) = -\frac{1}{1-\alpha}\left[\frac{L(s^t)}{K(s^{t-1})}\right]^{\alpha}\frac{U_l(s^t)}{U_c(s^t)}$$

$$(11) \qquad Q(s^t|s^{t-1}) = \beta^t \pi(s^t|s^{t-1})\frac{U_c(s^t)}{U_c(s^{t-1})}\frac{\bar{P}(s^{t-1})}{\bar{P}(s^t)}.$$

Notice that we have set $\beta = 1$ in the Euler equation for capital for convenience.

Before linearizing the equations in (5)-(8), we need some definitions. Let

$$(12) \qquad \begin{aligned} x_t &= \log(P(s^{t-1})) - \log(\bar{P}) \\ m_t &= \log(M(s^t)) - \log(\bar{M}) \\ p_t &= \log(\bar{P}(s^t)) - \log(\bar{P}) \\ y_t &= \log(Y(s^t)) - \log(\bar{Y}) \\ c_t &= \log(C(s^t)) - \log(\bar{C}) \\ \ell_t &= \log(L(s^t)) - \log(\bar{L}) \\ k_t &= \log(K(s^{t-1})) - \log(\bar{K}) \\ v_t &= \log(v(s^t)) - \log(\bar{v}) \\ \mu_t &= \log(\mu(s^t)) - \log(\bar{\mu}), \end{aligned}$$

where a '-' over the letter implies the steady state value of the variable. The variables in (12) are therefore logged deviations. Note that the growth rate μ_t can also be written as $m_t - m_{t-1}$. For the results below we assume that μ_t is an i.i.d. process.

Using the above definitions, we next derive the linearized pricing equation, money demand equation, Euler equation for capital, resource constraint, relation for the unit cost, and aggregate price level. These are given by

$$x_t = \frac{1}{2} E_{t-1}\{p_t + p_{t+1} + \upsilon_t + \upsilon_{t+1}\} \tag{13}$$

$$E_t\{a_1 c_t + a_2 c_{t+1} + a_3 \ell_t + a_4 \ell_{t+1} + a_5(m_t - p_t) + a_6(m_{t+1} - p_{t+1}) + a_7(p_t - p_{t+1})\} = 0 \tag{14}$$

$$E_t\{b_1 c_t + b_2 c_{t+1} + b_3 \ell_t + b_4 \ell_{t+1} + b_5(m_t - p_t) + b_6(m_{t+1} - p_{t+1}) + b_7 k_{t+1}\} = 0 \tag{15}$$

$$\bar{C} c_t + \bar{K} k_{t+1} - \bar{K}(1-\delta) k_t = \bar{Y}(\alpha k_t + (1-\alpha)\ell_t) = \bar{Y} y_t \tag{16}$$

$$\upsilon_t = (\alpha + \xi)\ell_t - \alpha k_t + \kappa c_t \tag{17}$$

$$p_t = \frac{1}{2} x_t + \frac{1}{2} x_{t-1}, \tag{18}$$

where $\bar{C}$, $\bar{K}$, and $\bar{Y}$ are the steady state values for consumption, capital, and output and $E_t \equiv E\{\cdot|\mu_t, \mu_{t-1}, \ldots\}$. The coefficients in the money demand equation are equal to

$$\begin{aligned}
a_1 &= -\sigma \bar{C}^{1-\kappa}/V - \kappa/\beta \\
a_2 &= \sigma \bar{C}^{1-\kappa}/V + \kappa \\
a_3 &= \sigma \omega_1 \bar{L}^{1+\xi}/V \\
a_4 &= -\sigma \omega_1 \bar{L}^{1+\xi}/V \\
a_5 &= -\sigma \omega_2 (\bar{M}/\bar{P})^{1-\nu}/V + \nu(1-\beta)/\beta \\
a_6 &= \sigma \omega_2 (\bar{M}/\bar{P})^{1-\nu}/V \\
a_7 &= 1,
\end{aligned} \tag{19}$$

where $\bar{L}$, $\bar{M}$, and $\bar{P}$ are steady state values for labor, money, and the price level, and $V = ((1-\sigma)U(\bar{C}, \bar{L}, \bar{M}/\bar{P})^{1-\sigma}$, The coefficients in the Euler equation for capital are equal to:

$$\begin{aligned}
b_1 &= -\sigma \bar{C}^{1-\kappa}/V - \kappa \\
b_2 &= \sigma \bar{C}^{1-\kappa}/V + \kappa - \delta\kappa
\end{aligned} \tag{20}$$

$$
\begin{aligned}
b_3 &= \sigma\omega_1\bar{L}^{1+\xi}/V \\
b_4 &= -\sigma\omega_1\bar{L}^{1+\xi}/V - \delta(1+\xi) \\
b_5 &= -\sigma\omega_2(\bar{M}/\bar{P})^{1-\nu}/V \\
b_6 &= \sigma\omega_2(\bar{M}/\bar{P})^{1-\nu}/V \\
b_7 &= \delta.
\end{aligned}
$$

Substituting (17) and (18) into the pricing equation (13) yields

$$
(21) \quad x_t = \frac{1}{2}x_{t-1} + \frac{1}{2}E_{t-1}x_{t+1} + E_{t-1}\{(\alpha+\xi)(\ell_t + \ell_{t+1}) - \alpha(k_t + k_{t+1}) + \kappa(c_t + c_{t+1})\}.
$$

From the linearized resource constraint and the production function, we have that

$$
(22) \quad \ell_t = \frac{1}{1-\alpha}(y_t - \alpha k_t)
$$

$$
(23) \quad c_t = \frac{\bar{Y}}{\bar{C}}y_t - \frac{\bar{K}}{\bar{C}}(k_{t+1} - (1-\delta)k_t).
$$

Substituting (22) and (23) into the pricing equation (21), the money demand equation (14), and the Euler equation for capital (15) gives three dynamic equations that depend only on output, the capital stock, prices, and the stock of money, i.e.,

$$
\begin{aligned}
(24) \quad x_t = {} & \frac{1}{2}x_{t-1} + \frac{1}{2}E_{t-1}x_{t+1} + E_{t-1}\{\left(\frac{\alpha+\xi}{1-\alpha} + \kappa\frac{\bar{Y}}{\bar{C}}\right)(y_t + y_{t+1}) \\
& - \left(\frac{\alpha(1+\xi)}{1-\alpha} - \kappa\frac{\bar{K}}{\bar{C}}(1-\delta)\right)(k_t + k_{t+1}) - \kappa\frac{\bar{K}}{\bar{C}}(k_{t+1} + k_{t+2})\}
\end{aligned}
$$

$$
\begin{aligned}
(25) \quad & E_t\{\left(a_1\frac{\bar{Y}}{\bar{C}} + a_3\frac{1}{1-\alpha}\right)y_t + \left(a_2\frac{\bar{Y}}{\bar{C}} + a_4\frac{1}{1-\alpha}\right)y_{t+1} + \left(a_1\frac{\bar{K}}{\bar{C}}(1-\delta) - a_3\frac{\alpha}{1-\alpha}\right)k_t \\
& - \left(a_1\frac{\bar{K}}{\bar{C}} - a_2\frac{\bar{K}}{\bar{C}}(1-\delta) + a_4\frac{\alpha}{1-\alpha}\right)k_{t+1} - a_2\frac{\bar{K}}{\bar{C}}k_{t+2} \\
& -\frac{1}{2}(a_5 - a_7)x_{t-1} - \frac{1}{2}(a_5 + a_6)x_t - \frac{1}{2}(a_6 + a_7)x_{t+1} + a_5 m_t + a_6 m_{t+1}\} = 0
\end{aligned}
$$

$$(26)\qquad E_t\{\left(b_1\frac{\bar{Y}}{\bar{C}}+b_3\frac{1}{1-\alpha}\right)y_t+\left(b_2\frac{\bar{Y}}{\bar{C}}+b_4\frac{1}{1-\alpha}\right)y_{t+1}+\left(b_1\frac{\bar{K}}{\bar{C}}(1-\delta)-b_3\frac{\alpha}{1-\alpha}\right)k_t$$
$$-\left(b_1\frac{\bar{K}}{\bar{C}}-b_2\frac{\bar{K}}{\bar{C}}(1-\delta)+b_4\frac{\alpha}{1-\alpha}-b_7\right)k_{t+1}-b_2\frac{\bar{K}}{\bar{C}}k_{t+2}$$
$$-\frac{1}{2}b_5x_{t-1}-\frac{1}{2}(b_5+b_6)x_t-\frac{1}{2}b_6x_{t+1}+b_5m_t+b_6m_{t+1}\}=0.$$

We substitute the guess in equations (1)-(3) into the linearized first order conditions in (24)-(26) and, after simplifying, the residuals reduce to the following set of equations

$$(27)\qquad 0=(\xi+\kappa)k_t$$

$$(28)\qquad 0=\left[\left(a_1\frac{\bar{Y}}{\bar{C}}+a_3\frac{1}{1-\alpha}\right)B+\left(a_2+a_4-a_1\frac{\bar{K}}{\bar{C}}\right)D\right]\mu_t+\left(1-\frac{1}{\beta}\right)(\kappa k_t-\nu(m_t-x_t))$$

$$(29)\qquad 0=\left[\left(b_1\frac{\bar{Y}}{\bar{C}}+b_3\frac{1}{1-\alpha}\right)B+\left(b_2+b_4-b_1\frac{\bar{K}}{\bar{C}}+b_7\right)D\right]\mu_t-\delta(\kappa+\xi)k_t.$$

The unknown coefficients B and D of (2) and (3) are found by setting the coefficients on μ_t in the above residuals to zero. With the coefficients on μ_t in the residuals equal to zero, the residuals are driven to zero as we drive ξ, κ, and ν to zero. This proves our claim.

We next show that the ratio of B to D is very large when we drive ξ and κ to zero. Note first that the steady state values $\bar{C}$, $\bar{K}$, $\bar{L}$, and $\bar{Y}$ satisfy the following expressions

$$(30)\qquad \bar{Y} = \bar{C}+\delta K$$

$$(31)\qquad \bar{Y} = \bar{K}^{\alpha}\bar{L}^{1-\alpha}$$

$$(32)\qquad \delta = \frac{\alpha}{1-\alpha}\frac{\bar{L}}{\bar{K}}\frac{\omega_1\bar{L}^{\xi}}{\bar{C}^{-\kappa}}$$

$$(33)\qquad \theta = \frac{1}{1-\alpha}\left[\frac{\bar{L}}{\bar{K}}\right]^{\alpha}\frac{\omega_1\bar{L}^{\xi}}{\bar{C}^{-\kappa}}$$

from the resource constraint, the production function, the Euler equation for capital, and the expression for unit costs. From (32), we have that

$$\omega_1 \bar{L}^{1+\xi} = \frac{\delta(1-\alpha)}{\alpha} \bar{K}\bar{C}^{-\kappa} \tag{34}$$

which can be substituted into a_3, a_4, b_3, and b_4. With some manipulation of equations (30)–(33), it is also easy to show that $\bar{C}/\bar{Y} = 1 - \alpha\theta$ and $\bar{K}/\bar{Y} = \alpha\theta/\delta$. Using these facts in addition to the expressions for the a's and b's in (19) and (20) with (34) already substituted in, the expression for B/D simplifies to:

$$\frac{B}{D} = 1 + \frac{\alpha}{\delta\left(\frac{1}{\theta} - 1\right)}, \tag{35}$$

when ξ and κ are set equal to 0. This is true in the limiting case with $\xi = 0$ and $\kappa = 0$ whether we use (28) or (29) to back out B/D.

Equation (35) shows that the response of output on impact is much larger than the response at all later dates assuming α is significantly larger than $\delta(\frac{1}{\theta} - 1)$. If we use the parameters for our baseline case with $N = 2$ and $T = 1/4$, we have $\alpha = 1/3$, $\delta = 1 - 0.92^{T/N}$, and $\theta = 0.9$. In this case $B/D = 290$. With a 1 percent increase in the log of the money supply, there is a $290 \times D$ percent increase in the log of output. In the next period, the output deviation is D percent. Therefore, the contract multiplier is tiny.