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NOTES: Is the Stock Market Overvalued?†

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These notes work through the details of the model reported in our paper. We describe the optimization problems of the various agents and derive the first-order conditions of these problems. We then describe how to compute equilibria and asset prices.

1. The Optimization Problems

The household's problem: choose sequences of consumptions, investments, and hours of leisure/work that solve

$$\max E\left[\sum_{t=0}^{\infty} \beta^t N_t U(c_t, l_t) \mid X_0\right]$$

$$\begin{aligned}
\text{subject to}\quad & N_t\left[(1+\tau_{c,t})c_t + x_{1m,t} + x_{1u,t} + x_{2,t}\right] \\
&\quad = N_t\Big[r_{1m,t}k_{1m,t} + r_{1u,t}k_{1u,t} + r_{2,t}k_{2,t} + w_t n_t \\
&\qquad\quad - \tau_{1,t}\left[(r_{1m,t} - \delta_{1m} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right] \\
&\qquad\quad - \tau_{2,t}\left[(r_{2,t} - \delta_2 - \tau_{2k,t})k_{2,t}\right] - \tau_{n,t} w_t n_t \\
&\qquad\quad - \tau_{1k,t}k_{1m,t} - \tau_{2k,t}k_{2,t} + T_t\Big] \\
& N_{t+1} = (1+\eta)N_t \\
& n_t + l_t = 1 \\
& (1+\eta)k_{i,t+1} = (1-\delta_i)k_{i,t} + x_{i,t} - \varphi\left(\frac{x_{i,t}}{k_{i,t}}\right)k_{i,t}, \quad i = 1m, 1u, 2
\end{aligned}$$

where t indexes time, N is the size of the household, c is consumption, l is time allocated to leisure, x denotes investment, r denotes the rental rate on capital, k denotes a capital stock, w is the wage rate, n is time at work, τ denotes a proportional tax rate, δ denotes a depreciation rate, and T is transfers from the government. Households choose between investing in the corporate sector (sector 1) and the noncorporate sector (sector 2). Some investment in the corporate sector is "measured" by national accountants and some is "unmeasured" or intangible capital. We denote measured investment in the corporate sector in period t as $x_{1m,t}$. Similarly, unmeasured investment in the corporate sector at date t is $x_{1u,t}$. Investment in the noncorporate sector at date t is $x_{2,t}$. All investment in the noncorporate sector is assumed to be measured by national accountants. We use similar subscripting notation for rental rates, capital stocks, and depreciation rates. Households face a number of taxes – all of which may be stochastically changing. They pay consumption taxes at rate τ_c. They pay corporate and noncorporate income taxes at rates τ_1 and τ_2, respectively. They pay labor income taxes at rate τ_n. Finally, they pay taxes on corporate and noncorporate property at rates τ_{1k} and τ_{2k} respectively. All variables are in per-capita terms. Therefore, n is the fraction of family members working and l is the fraction enjoying leisure.

Problem for a firm in sector 1: choose capital and labor to solve

$$\max \; p_1 y_1 - r_{1m} k_{1m} - r_{1u} k_{1u} - w n_1$$
$$\text{subject to} \;\; y_1 = k_{1m}^{\phi_m} k_{1u}^{\phi_u} (z n_1)^{1-\phi_m-\phi_u}$$

where price p_1 is taken as given and log z, ϕ_m, and ϕ_n are exogenous stochastic processes.

Problem for a firm in sector 2: choose capital and labor to solve

$$\max \; p_2 y_2 - r_2 k_2 - w n_2$$
$$\text{subject to} \;\; y_2 = k_2^{\theta} (z n_2)^{1-\theta}$$

where price p_2 is taken as given and log z and θ are exogenous stochastic processes.

The final goods firm's problem: choose intermediate and final outputs to solve

$$\max \; y - p_1 y_1 - p_2 y_2$$
$$\text{subject to} \;\; y = A(\mu y_1^{\rho} + (1-\mu) y_2^{\rho})^{\frac{1}{\rho}}.$$

The government budget constraint:

$$g_t + T_t = \tau_{c,t} c_t + \tau_{1,t} \left[(r_{1m,t} - \delta_{1m} - \tau_{1k,t}) k_{1m,t} + r_{1u,t} k_{1u,t} - x_{1u,t} \right]$$
$$+ \tau_{2,t} \left[(r_{2,t} - \delta_2 - \tau_{2k,t}) k_{2,t} \right] + \tau_{n,t} w_t n_t + \tau_{1k,t} k_{1m,t} - \tau_{2k,t} k_{2,t}$$

Additionally, in equilibrium, we need:

$$n_t = n_{1,t} + n_{2,t}$$

$$y_t = c_t + x_{1m,t} + x_{1u,t} + x_{2,t} + g_t.$$

2. The First-order Conditions

The household. The Lagrangian for the problem is given by

$$
\begin{aligned}
\mathcal{L} = E\Big[& \beta^t(1+\eta)^t U(c_t, l_t) \\
& + \lambda_t \big\{ r_{1m,t}k_{1m,t} + r_{1u,t}k_{1u,t} + r_{2,t}k_{2,t} + w_t(1-l_t) \\
& \qquad - \tau_{1,t}\left[(r_{1m,t} - \delta_{1m} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right] \\
& \qquad - \tau_{2,t}\left[(r_{2,t} - \delta_2 - \tau_{2k,t})k_{2,t}\right] - \tau_{n,t}w_t(1-l_t) \\
& \qquad - \tau_{1k,t}k_{1m,t} - \tau_{2k,t}k_{2,t} + T_t \big\} \\
& \qquad - (1+\tau_{c,t})c_t - x_{1m,t} - x_{1u,t} - x_{2,t} \big\} \\
& + \mu_{1m,t}\big\{(1-\delta_{1m})k_{1m,t} + x_{1m,t} - \varphi(x_{1m,t}/k_{1m,t})k_{1m,t} - (1+\eta)k_{1m,t+1}\big\} \\
& + \mu_{1u,t}\big\{(1-\delta_{1u})k_{1u,t} + x_{1u,t} - \varphi(x_{1u,t}/k_{1u,t})k_{1u,t} - (1+\eta)k_{1u,t+1}\big\} \\
& + \mu_{2,t}\big\{(1-\delta_2)k_{2,t} + x_{2,t} - \varphi(x_{2,t}/k_{2,t})k_{2,t} - (1+\eta)k_{2,t+1}\big\}\Big]
\end{aligned}
$$

Taking derivatives with respect to c_t, l_t, $x_{1m,t}$, $x_{2,t}$, $x_{1u,t}$, $k_{1m,t+1}$, $k_{2,t+1}$, $k_{1u,t+1}$,

$$0 = \tilde{\beta}^t U_{ct} - \lambda_t(1+\tau_{ct})$$

$$0 = \tilde{\beta}^t U_{lt} - \lambda_t w_t(1-\tau_{n,t})$$

$$0 = -\lambda_t + \mu_{1m,t}(1-\varphi'_{1m,t})$$

$$0 = -\lambda_t + \mu_{2,t}(1-\varphi'_{2,t})$$

$$0 = \lambda_t\tau_{1,t} - \lambda_t + \mu_{1u,t}(1-\varphi'_{1u,t})$$

$$
\begin{aligned}
0 = {} & \lambda_{t+1}\left(r_{1m,t+1} - \tau_{1,t+1}(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1}) - \tau_{1k,t+1}\right) \\
& + \mu_{1m,t+1}\left(1 - \delta_{1m} - \varphi_{1m,t+1} + \varphi'_{1m,t+1}x_{1m,t+1}/k_{1m,t+1}\right) - (1+\eta)\mu_{1m,t}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \lambda_{t+1}\left(r_{2,t+1} - \tau_{2,t+1}(r_{2,t+1} - \delta_2 - \tau_{2k,t+1}) - \tau_{2k,t+1}\right) \\
& + \mu_{2,t+1}\left(1 - \delta_2 - \varphi_{2,t+1} + \varphi'_{2,t+1}x_{2,t+1}/k_{2,t+1}\right) - (1+\eta)\mu_{2,t}
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \lambda_{t+1}\left(r_{1u,t+1} - \tau_{1,t+1}r_{1u,t+1}\right) \\
& + \mu_{1u,t+1}\left(1 - \delta_{1u} - \varphi_{1u,t+1} + \varphi'_{1u,t+1}x_{1u,t+1}/k_{1u,t+1}\right) - (1+\eta)\mu_{1u,t}
\end{aligned}
$$

The multipliers in this case are

$$\lambda_t = \tilde{\beta}^t U_{ct}/(1+\tau_{c,t})$$

$$\mu_{1m,t} = \tilde{\beta}^t U_{ct}/(1+\tau_{c,t})/(1-\varphi'_{1m,t})$$

$$\mu_{2,t} = \tilde{\beta}^t U_{ct}/(1+\tau_{c,t})/(1-\varphi'_{2,t})$$

$$\mu_{1u,t} = \tilde{\beta}^t U_{ct}(1-\tau_{1,t})/(1+\tau_{ct})/(1-\varphi'_{1u,t})$$

Using these expressions for the multipliers, we have the following first-order conditions,

$$\frac{U_{lt}}{U_{ct}} = w_t \frac{1-\tau_{n,t}}{1+\tau_{c,t}}$$

$$\frac{U_{ct}}{1+\tau_{c,t}} = \beta E_t\left[\frac{U_{ct+1}}{1+\tau_{c,t+1}}\left\{\text{return}_{1m,t+1}\right\}\right]$$

$$\text{return}_{1m,t+1} = (1-\varphi'_{1m,t})\Bigg\{r_{1m,t+1} - \tau_{1,t+1}(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1}) - \tau_{1k,t+1}$$
$$+ \frac{1}{1-\varphi'_{1m,t+1}}\left[1-\delta_{1m} - \varphi_{1m,t+1} + \varphi'_{1m,t+1}\frac{x_{1m,t+1}}{k_{1m,t+1}}\right]\Bigg\}$$

$$\frac{U_{ct}}{1+\tau_{c,t}} = \beta E_t\left[\frac{U_{ct+1}}{1+\tau_{c,t+1}}\left\{\text{return}_{2,t+1}\right\}\right]$$

$$\text{return}_{2,t+1} = (1-\varphi'_{2,t})\Bigg\{r_{2,t+1} - \tau_{2,t+1}(r_{2,t+1} - \delta_2 - \tau_{2k,t+1}) - \tau_{2k,t+1}$$
$$+ \frac{1}{1-\varphi'_{2,t+1}}\left[1-\delta_2 - \varphi_{2,t+1} + \varphi'_{2,t+1}\frac{x_{2,t+1}}{k_{2,t+1}}\right]\Bigg\}$$

$$\frac{U_{ct}}{1+\tau_{c,t}} = \beta E_t\left[\frac{U_{ct+1}}{1+\tau_{c,t+1}}\left\{\text{return}_{1u,t+1}\right\}\right]$$

$$\text{return}_{1u,t+1} = (1-\varphi'_{1u,t})\left(\frac{1-\tau_{1,t+1}}{1-\tau_{1,t}}\right)\Bigg\{r_{1u,t+1}$$
$$+ \frac{1}{1-\varphi'_{1u,t+1}}\left[1-\delta_{1u} - \varphi_{1u,t+1} + \varphi'_{1u,t+1}\frac{x_{1u,t+1}}{k_{1u,t+1}}\right]\Bigg\}$$

where $X_t = [k_{1m,t},\ k_{1u,t},\ k_{2,t},\ s_t]'$ and $s_t = [\omega_t,\ \tau_{c,t},\ \tau_{1,t},\ \tau_{2,t},\ \tau_{n,t},\ g_t,\ \phi_{m,t},\ \phi_{u,t},\ \theta_t,\ \tau_{1k,t},$ $\tau_{2k,t}]'$. The variable ω is the shock to technology z which is a unit-root process,

$$\log\ z_{t+1} = \log\ z_t + \log(1+\gamma) + \omega_{t+1}$$

which grows at rate γ.

The firms. The rental rates and wage rates are simply marginal value products derived from the problem of an intermediate good firm. The relative prices p_1 and p_2 are derived from the final good firm's problem. These prices are given by

$$\begin{aligned}
r_{1m} &= p_1\phi_m k_{1m}^{\phi_m-1} k_{1u}^{\phi_u} (zn_1)^{1-\phi_m-\phi_u} \\
r_{1u} &= p_1\phi_u k_{1m}^{\phi_m} k_{1u}^{\phi_u-1} (zn_1)^{1-\phi_m-\phi_u} \\
w &= p_1(1-\phi_m-\phi_u) k_{1m}^{\phi_m} k_{1u}^{\phi_u} n_1^{-\phi_m-\phi_u} z^{1-\phi_m-\phi_u} \\
w &= p_2(1-\theta) k_2^{\theta} n_2^{-\theta} z^{1-\theta} \\
p_1 &= A^{\rho}\mu(y/y_1)^{1-\rho} \\
p_2 &= A^{\rho}(1-\mu)(y/y_1)^{1-\rho}
\end{aligned}$$

3. Equilibrium decision functions

Suppose that we have computed the following decision functions: $x_{1m,t} = x_{1m}(k_{1m,t},\ k_{1u,t},$ $k_{2,t},\ s_t)$, $x_{1u,t} = x_{1u}(k_{1m,t}, k_{1u,t}, k_{2,t}, s_t)$ and $n_{1,t} = n_1(k_{1m,t}, k_{1u,t}, k_{2,t}, s_t)$. With these, we can back out everything else.

- Step 1. Set pre-tax wages paid in the two sectors:

$$p_1(1-\phi_m-\phi_u) k_{1m}^{\phi_m} k_{1u}^{\phi_u} n_1^{-\phi_m-\phi_u} z^{1-\phi_m-\phi_u} = p_2(1-\theta) k_2^{\theta} n_2^{-\theta} z^{1-\theta}$$

  which implies a price ratio of

$$\frac{p_1}{p_2} = \frac{(1-\theta) k_2^{\theta} n_2^{-\theta} z^{1-\theta}}{(1-\phi_m-\phi_u) k_{1m}^{\phi_m} k_{1u}^{\phi_u} n_1^{-\phi_m-\phi_u} z^{1-\phi_m-\phi_u}}.$$

- Step 2. Equate price ratio from Step 1 to price ratio found from final goods producer's problem:

$$\frac{(1-\theta)k_2^{\theta}n_2^{-\theta}z^{1-\theta}}{(1-\phi_m-\phi_u)k_{1m}^{\phi_m}k_{1u}^{\phi_u}n_1^{-\phi_m-\phi_u}z^{1-\phi_m-\phi_u}} = \frac{\mu y_1^{\rho-1}}{(1-\mu)y_2^{\rho-1}}$$

$$= \frac{\mu(k_{1m}^{\phi_m}k_{1u}^{\phi_u}(zn_1)^{1-\phi_m-\phi_u})^{\rho-1}}{(1-\mu)(k_2^{\theta}(zn_2)^{1-\theta})^{\rho-1}}.$$

This gives us an expression for n_2:

$$n_2 = \left[\left(\frac{(1-\theta)(1-\mu)}{(1-\phi_m-\phi_u)\mu}\right)\left(\frac{k_2^{\theta\rho}z^{-\theta\rho}}{k_{1m}^{\phi_m\rho}k_{1u}^{\phi_u\rho}z^{(-\phi_m-\phi_u)\rho}}\right)n_1^{1-\rho(1-\phi_m-\phi_u)}\right]^{\frac{1}{1-\rho(1-\theta)}}.$$

- Step 3. With the states $k_{1m,t}$, $k_{1u,t}$, $k_{2,t}$, and s_t and the hours $n_{1,t}$ and $n_{2,t}$, we can get output levels, prices, and leisure:

$$\begin{aligned}
y_{1,t} &= k_{1m,t}^{\phi_{m,t}}k_{1u,t}^{\phi_{1u,t}}(z_t n_{1,t})^{1-\phi_{m,t}-\phi_{u,t}}\\
y_{2,t} &= k_{2,t}^{\theta_t}(z_t n_{2,t})^{1-\theta_t}\\
y_t &= A(\mu y_{1,t}^{\rho} + (1-\mu)y_{2,t}^{\rho})^{\frac{1}{\rho}}\\
p_{1,t} &= A^{\rho}\mu(y_t/y_{1,t})^{1-\rho}\\
p_{2,t} &= A^{\rho}(1-\mu)(y_t/y_{2,t})^{1-\rho}\\
r_{1m,t} &= p_{1,t}\phi_{m,t}y_{1,t}/k_{1m,t}\\
r_{1u,t} &= p_{1,t}\phi_{u,t}y_{1,t}/k_{1u,t}\\
r_{2,t} &= p_{2,t}\theta_t y_{2,t}/k_{2,t}\\
w_t &= p_{1,t}(1-\phi_{1m,t}-\phi_{1u,t})y_{1,t}/n_{1,t}\\
l_t &= 1-n_{2,t}-n_{1,t}.
\end{aligned}$$

- Step 4. With leisure l_t, the wage rate w_t, and the tax on hours $\tau_{n,t}$, we can back out consumption from

$$U_c(c_t,l_t) = U_c(c_t,l_t)w_t(1-\tau_{n,t})/(1+\tau_{c,t}).$$

- Step 5. With corporate investments $x_{1m,t}$, $x_{1u,t}$, consumption c_t, total output y_t, and government spending g_t, we can back out total investment and investment in noncorporate capital:

$$x_t = y_t - c_t - g_t$$

$$x_{1,t} = x_{1m,t} + x_{1u,t}$$

$$x_{2,t} = x_t - x_{1,t}.$$

- Step 6. With the capital stocks and investments, we can determine next period's capital stocks:

$$k_{i,t+1} = [(1-\delta_i)k_{i,t} + x_{i,t} - \varphi(x_{i,t}/k_{i,t})k_{i,t}]/(1+\eta), \qquad i = 1m, 1u, 2.$$

We now need to compute $x_{1m,t}$, $x_{1u,t}$, and $n_{1,t}$. First, we transform the first-order equations so that all variables are stationary. The household dynamic Euler equations look as before since $1+\eta$ will not appear there. For this problem, we have three since there are three capital stocks. The equations for capital accumulation are now

$$\tilde{k}_{i,t+1} = \left[(1-\delta_i)\tilde{k}_{i,t} + \tilde{x}_{i,t} - \varphi(\tilde{x}_{i,t}/\tilde{k}_{i,t})\tilde{k}_{i,t}\right] e^{-\omega_{t+1}}/[(1+\gamma)(1+\eta)], \qquad i = 1m, 1u, 2.$$

We turn next to solving our system of difference equations. We use the static first-order conditions to eliminate n_{2t}, $\tilde{y}_{1,t}$, $\tilde{y}_{2,t}$, $\tilde{y}_t$, $p_{1,t}$, $p_{2,t}$, $r_{1,t}$, $r_{2,t}$, $\tilde{w}_t$, l_t, $\tilde{c}_t$, $\tilde{x}_t$, and $\tilde{x}_{2,t}$ in the dynamic Euler equations and the capital accumulation equations. We then linearize these equations and get a system of equations that look like

$$E_t\left\{\Psi_1\begin{bmatrix}\hat{k}_{1m,t+1}\\ \hat{k}_{1u,t+1}\\ \hat{k}_{2,t+1}\end{bmatrix} + \Psi_2\begin{bmatrix}\hat{x}_{1m,t+1}\\ \hat{x}_{1u,t+1}\\ \hat{n}_{1,t+1}\end{bmatrix} + \Psi_3\begin{bmatrix}\hat{k}_{1m,t}\\ \hat{k}_{1u,t}\\ \hat{k}_{2,t}\end{bmatrix} + \Psi_4\begin{bmatrix}\hat{x}_{1m,t}\\ \hat{x}_{1u,t}\\ \hat{n}_{1,t}\end{bmatrix} + \Psi_5\hat{s}_{t+1} + \Psi_6\hat{s}_t\right\} = 0$$

$$\Phi_1\begin{bmatrix}\hat{k}_{1m,t+1}\\ \hat{k}_{1u,t+1}\\ \hat{k}_{2,t+1}\end{bmatrix} + \Phi_2\begin{bmatrix}\hat{x}_{1m,t+1}\\ \hat{x}_{1u,t+1}\\ \hat{n}_{1,t+1}\end{bmatrix} + \Phi_3\begin{bmatrix}\hat{k}_{1m,t}\\ \hat{k}_{1u,t}\\ \hat{k}_{2,t}\end{bmatrix} + \Phi_4\begin{bmatrix}\hat{x}_{1m,t}\\ \hat{x}_{1u,t}\\ \hat{n}_{1,t}\end{bmatrix} + \Phi_5\hat{s}_{t+1} + \Phi_6\hat{s}_t = 0$$

where $\hat{}$ denotes stationary variables that have been logged and demeaned, e.g., $\hat{k}_{1m,t+1} = \log \tilde{k}_{1m,t+1} - \log k_{1m}$. All variables in $\hat{s}_t$ are in levels – with means subtracted – except government spending which is logged and demeaned. We assume that the process for $\hat{s}_t$ is given by

$$\hat{s}_{t+1} = P\hat{s}_t + \epsilon_{t+1}$$

where ϵ is a mean 0, serially uncorrelated vector of shocks.

We are looking for a solution of the form:

$$\begin{bmatrix} \hat{x}_{1m,t} \\ \hat{x}_{1u,t} \\ \hat{n}_{1,t} \end{bmatrix} = A \begin{bmatrix} \hat{k}_{1m,t} \\ \hat{k}_{1u,t} \\ \hat{k}_{2,t} \end{bmatrix} + B\hat{s}_t.$$

Since the system of equations is linear, we can ignore the stochastic terms while solving A. This matrix is found by computing the eigenvalues and eigenvectors of the following matrix:

$$-\begin{bmatrix} \Phi_1 & 0 \\ \Psi_1 & \Psi_2 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_3 & \Phi_4 \\ \Psi_3 & \Psi_4 \end{bmatrix}$$

(or generalized eigenvalues if Ψ_2 is not invertible). Notice that a 0 is written in for Φ_2 since the capital accumulation equations don't depend on decision variables in $t+1$. Suppose this matrix is equal to $V \Lambda V^{-1}$ where V are eigenvectors and Λ are eigenvalues. Assuming the eigenvalues are ordered so that the first two are inside the unit circle, we have our solution:

$$A = V_{21} V_{11}^{-1}$$

where V_{11} and V_{21} are 2×2 partitions of V (i.e., the upper left and lower left partitions, respectively).

Given A, we can compute B using the method of undetermined coefficients. First note that

$$\begin{bmatrix} \hat{k}_{1m,t+1} \\ \hat{k}_{1u,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} = -\Phi_1^{-1}(\Phi_3 + \Phi_4 A) \begin{bmatrix} \hat{k}_{1m,t} \\ \hat{k}_{1u,t+1} \\ \hat{k}_{2,t} \end{bmatrix} - \Phi_1^{-1}\Phi_5 \hat{s}_{t+1} - \Phi_1^{-1}(\Phi_6 + \Phi_4 B)\hat{s}_t.$$

Next note that B must be set so that

$$
\begin{aligned}
0 = E_t\Bigg\{ & \Psi_1 \begin{bmatrix} \hat{k}_{1m,t+1} \\ \hat{k}_{1u,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Psi_2 \begin{bmatrix} \hat{x}_{1m,t+1} \\ \hat{x}_{1u,t+1} \\ \hat{n}_{1,t+1} \end{bmatrix} + \Psi_3 \begin{bmatrix} \hat{k}_{1m,t} \\ \hat{k}_{1u,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Psi_4 \begin{bmatrix} \hat{x}_{1m,t} \\ \hat{x}_{1u,t} \\ \hat{n}_{1,t} \end{bmatrix} + \Psi_5 \hat{s}_{t+1} + \Psi_6 \hat{s}_t \Bigg\} \\
= E_t\Bigg\{ & \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_3 + \Phi_4 A) + \Psi_3 + \Psi_4 A\right] \left[\, \hat{k}_{1m,t} \quad \hat{k}_{1u,t} \quad \hat{k}_{2,t} \,\right]' \\
& + \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 + \Psi_5 + \Psi_2 B\right] \hat{s}_{t+1} \\
& \quad + \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B\right] \hat{s}_t \Bigg\} \\
= \; & \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_3 + \Phi_4 A) + \Psi_3 + \Psi_4 A\right] \left[\, \hat{k}_{1m,t} \quad \hat{k}_{1u,t} \quad \hat{k}_{2,t} \,\right]' \\
& + \Big[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P + \Psi_5 P + \Psi_2 B P \\
& \quad - (\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B\Big] \hat{s}_t
\end{aligned}
$$

is satisfied. For this to be satisfied, we need the coefficient on $\hat{s}_t$ equal to zero. Thus, we have 44 equations in 44 unknowns (i.e., the elements of B):

$$
\begin{aligned}
0_{4\times 11} = & -(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P + \Psi_5 P + \Psi_2 B P \\
& - (\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B.
\end{aligned}
$$

This can be written as a simple linear system in $vec(B)$ if we apply the standard formula: $vec(ABC) = [C' \otimes A]vec(B)$. Doing this we get:

$$
\begin{aligned}
vec(B) = & \left\{[P' \otimes \Psi_2] + [I \otimes -(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_4] + [I \otimes \Psi_4]\right\}^{-1} \\
& vec((\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P - \Psi_5 P + (\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_6 - \Psi_6)
\end{aligned}
$$

4. Steady State Equations

For our computation, we work with the preferences

$$U(c,l) = \frac{(c\,l^{\psi})^{1-\sigma}}{1-\sigma}.$$

When we write this as a function of transformed variables we replace c with $z\tilde{c}$.

For our linear approximation, we need to compute the steady state of the transformed economy. The nonlinear equations to be solved are:

$$1 = \beta(1+\gamma)^{-\sigma}\left[(r_{1m} - \delta_{1m} - \tau_{1k})(1-\tau_1) + 1\right]$$

$$1 = \beta(1+\gamma)^{-\sigma}\left[r_{1u} - \delta_{1u} + 1\right]$$

$$1 = \beta(1+\gamma)^{-\sigma}\left[(r_2 - \delta_2 - \tau_{2k})(1-\tau_2) + 1\right]$$

$$\psi\tilde{c}/l = (1-\tau_n)w/(1+\tau_c)$$

$$\tilde{k}_i = [(1-\delta_1)\tilde{k}_i + \tilde{x}_i]/(1+\gamma), \quad i = 1m, 1u, 2$$

$$\tilde{y} = \tilde{c} + \tilde{x}_{1m}\tilde{x}_{1u} + \tilde{x}_2 + \tilde{g}$$

$$\tilde{y} = A(\mu\tilde{y}_1^{\rho} + (1-\mu)\tilde{y}_2^{\rho})^{\frac{1}{\rho}}$$

$$\tilde{y}_1 = (k_{1m})^{\phi_m}(k_{1u})^{\phi_u} n_1^{1-\phi_m-\phi_u}$$

$$\tilde{y}_2 = k_2^{\theta} n_2^{1-\theta}$$

$$p_1 = A^{\rho}\mu(\tilde{y}/\tilde{y}_1)^{1-\rho}$$

$$p_i = A^{\rho}(1-\mu)(\tilde{y}/\tilde{y}_2)^{1-\rho}$$

$$p_1(1-\phi_m-\phi_u)(k_{1m})^{\phi_m}(k_{1u})^{\phi_u} n_1^{-\phi_m-\phi_u} = p_2(1-\theta)k_2^{\theta} n_2^{-\theta}$$

which is 14 equations in 14 unknowns: $\tilde{k}_{1m}$, $\tilde{k}_{1u}$, $\tilde{k}_2$, n_1, n_2, p_1, p_2, $\tilde{c}$, $\tilde{x}_{1m}$, $\tilde{x}_{1u}$, $\tilde{x}_2$, $\tilde{y}$, $\tilde{y}_1$, and $\tilde{y}_2$ (once definitions for r's and w's plugged in).

5. Test Cases

We compare the results to those computed in "A Progress Report on Business Cycle Models" (QR, Fall 1994) where technology is assumed to be an AR(1) process. To make things comparable we set the serial correlation coefficient near 1 in the older programs. We set $\mu = 1/2$ and $\rho = 1$ in the new program. These are one-sector versions of the model.

6. Computing Returns

We will follow the approach in Jermann (JME, 1998) who uses a log-linear approximation (as done above) to compute the law of motion for the capital stocks and shocks and nonlinear asset-pricing formulas to compute returns. When he compares his results to those found with nonlinear solution methods, he finds equity premia that are very close.

Define the vector X_t as $X_t = [\hat{k}_{1m,t}, \hat{k}_{1u,t}, \hat{k}_{2,t}, \hat{s}_t]$. From our earlier algebra, we have

$$X_{t+1} = A_o X_t + B_o \epsilon_{t+1}$$

where we will assume that ϵ is a multivariate normal i.i.d. shock. Let λ_t be the logarithm of the marginal utility – after transforming variables – with the mean subtracted off. The expression for λ_t is found as follows

$$\begin{aligned}
\lambda_t &= \ln\left(\tilde{c}_t^{-\sigma} l_t^{\psi(1-\sigma)}\right) - \ln\left(\tilde{c}^{-\sigma} l^{\psi(1-\sigma)}\right) \\
&= -\sigma \hat{c}_t + \psi(1-\sigma)\hat{l}_t \\
&\simeq \alpha_1 \begin{bmatrix} \hat{k}_{1m,t} \\ \hat{k}_{1u,t} \\ \hat{k}_{2,t} \end{bmatrix} + \alpha_2 \begin{bmatrix} \hat{x}_{1m,t} \\ \hat{x}_{1u,t} \\ \hat{n}_{1,t} \end{bmatrix} + \alpha_3 \hat{s}_t \\
&= (\alpha_1 + \alpha_2 A) \begin{bmatrix} \hat{k}_{1m,t} \\ \hat{k}_{1u,t} \\ \hat{k}_{2,t} \end{bmatrix} + (\alpha_2 B + \alpha_3)\hat{s}_t \\
&= [\alpha_1 + \alpha_2 A, \alpha_2 B + \alpha_3]' X_t \\
&\equiv \alpha' X_t.
\end{aligned}$$

We want to use this formula to compute an analytical expression for the expected gross risk-free rate, $ER_{f,t}$, where $R_{f,t}$ is given by:

$$
\begin{aligned}
R_{f,t} &= \left[\beta E_t \frac{U_c(c_{t+1}, l_{t+1})}{U_c(c_t, l_t)}\right]^{-1} \\
&= \left[\beta E_t \frac{c_{t+1}^{-\sigma} l_{t+1}^{\psi(1-\sigma)}}{c_t^{-\sigma} l_t^{\psi(1-\sigma)}}\right]^{-1} \\
&= \left[\beta(1+\gamma)^{-\sigma} E_t e^{-\sigma\omega_{t+1}+\lambda_{t+1}-\lambda_t}\right]^{-1}
\end{aligned}
$$

Since ϵ is normal, X is normal and λ is normal. Using the fact that $Ee^x = e^{1+\gamma+\sigma^2/2}$ when x is normally distributed with mean $1+\gamma$ and variance σ^2, we have

$$
\begin{aligned}
ER_t^f &= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[E_t \exp(-\sigma\omega_{t+1} + \lambda_{t+1} - \lambda_t)\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[E_t \exp(-\sigma\rho_\omega\omega_t - \sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - \lambda_t + E_t\lambda_{t+1} - E_t\lambda_{t+1})\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\omega\omega_t + E_t\lambda_{t+1} - \lambda_t) E_t \exp(-\sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - E_t\lambda_{t+1})\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\omega\omega_t + E_t\lambda_{t+1} - \lambda_t) \exp(\frac{1}{2}\mathrm{var}_t[-\sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}])\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\omega\omega_t + E_t\lambda_{t+1} - \lambda_t) \exp(\frac{1}{2}\mathrm{var}[-\sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}])\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E \exp(\sigma\rho_\omega\omega_t - E_t\lambda_{t+1} + \lambda_t) \exp(-\frac{1}{2}\mathrm{var}[-\sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}]) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\mathrm{var}[\sigma\rho_\omega\omega_t - E_t\lambda_{t+1} + \lambda_t] - \frac{1}{2}\mathrm{var}[-\sigma\epsilon_{\omega,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}]\right) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\mathrm{var}([\sigma\rho_\omega e' + \alpha'(I - A_o)]X_t) - \frac{1}{2}\mathrm{var}([-\sigma e' + \alpha']B_o\epsilon_{t+1})\right) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\mu' V_x \mu - \frac{1}{2}\kappa' B_o B_o' \kappa\right)
\end{aligned}
$$

where e is a 9×1 vector of zeros except for $e(3) = 1$ and

$$\mu = [\sigma\rho_\omega e' + \alpha'(I - A_o)]'$$

$$\kappa = [-\sigma e' + \alpha']'$$

$$V_x = A_o V_z A_o' + B_o B_o'.$$

To compute the expected return on corporate equities, we need to simulate the model and compute

$$r^e_{t,t+1} = \frac{[N_{t+2}V_{t+1}(X_{t+1}) + d_{t+1}(X_{t+1})N_{t+1}]}{V_t(X_t)N_{t+1}}.$$

Therefore, we want to compute the following objects:

- Per-capita Value added of sector 1: $p_{1,t}y_{1,t}$

  $=$ NIPA Corporate VA $-$ Corporate-non-property-tax IBT $+x_{1u}$

  (1) Wages: $w_t n_{1,t}$ (=NIPA corporate wages)

  (2) Measured depreciation: $\delta_{1m}k_{1m,t}$ (=NIPA corporate depreciation)

  (3) Measured property tax $\tau_{1k,t}k_{1m,t}$ (= NIPA, part of corporate IBT):

  (4) Profits: $(r_{1m,t} - \delta_{1m} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t}$ (=NIPA profits + $x_{1u,t}$)

  Tax: $\tau_{1,t}\left[(r_{1m,t} - \delta_{1m,t} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right]$ (=NIPA taxes)

  Retained earnings: $x_{1m,t} - \delta_{1m}k_{1m,t} + x_{1u,t}$ (=NIPA r.e. + $x_{1u,t}$)

  Dividends: residual (=NIPA dividends)

- Per-capita Dividends:

$$\begin{aligned}
d_t &= p_{1,t}y_{1,t} - w_t n_{1,t} - \tau_{1k,t}k_{1m,t} \\
&\quad - \tau_{1,t}\left[(r_{1m,t} - \delta_{1m,t} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right] - x_{1m,t} - x_{1u,t} \\
&= r_{1m,t}k_{1m,t} + r_{1u,t}k_{1u,t} - \tau_{1k,t}k_{1m,t} \\
&\quad - \tau_{1,t}\left[(r_{1m,t} - \delta_{1m} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right] \\
&\quad - (1+\eta)k_{1m,t+1} + k_{1m,t} - \delta_{1m}k_{1m,t} - \varphi_{1m,t}k_{1m,t} - x_{1u,t} \\
&= (1 - \tau_{1,t})\left[(r_{1m,t} - \delta_{1m} - \tau_{1k,t})k_{1m,t} + r_{1u,t}k_{1u,t} - x_{1u,t}\right] \\
&\quad - (1+\eta)k_{1m,t+1} + k_{1m,t}(1 - \varphi_{1m,t})
\end{aligned}$$

- Per-capita value of corporate equities:

$$V_t = \frac{k_{1m,t+1}}{1 - \varphi'_{1m,t+1}} + (1 - \tau_{1,t})\frac{k_{1u,t+1}}{1 - \varphi'_{1u,t+1}}$$

- Return on corporate equities:

$$r^e_{t,t+1} = \Big[(1+\eta)k_{1m,t+2}/(1-\varphi'_{1m,t+1}) + (1+\eta)(1-\tau_{1,t+1})k_{1u,t+2}/(1-\varphi'_{1u,t+1})$$

$$+ (1-\tau_{1,t+1})(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1})k_{1m,t+1}$$

$$+ (1-\tau_{1,t+1})(r_{1u,t+1}k_{1u,t+1} - x_{1u,t+1})$$

$$- (1+\eta)k_{1m,t+2} + k_{1m,t+1}(1-\varphi_{1m,t+1})\Big]$$

$$/\left(k_{1m,t+1}/(1-\varphi'_{1m,t}) + (1-\tau_{1,t})k_{1u,t+1}/(1-\varphi'_{1u,t})\right) - 1$$

$$= \frac{\text{return}_{1m,t+1}\frac{k_{1m,t+1}}{1-\varphi'_{1m,t}} + \text{return}_{1u,t+1}\frac{(1-\tau_{1,t})k_{1m,t+1}}{1-\varphi'_{1u,t}}}{\frac{k_{1m,t+1}}{1-\varphi'_{1m,t}} + \frac{(1-\tau_{1,t})k_{1m,t+1}}{1-\varphi'_{1u,t}}}$$

If there are no costs of adjusting capital, then $\varphi = 0$ and $\varphi' = 0$. In this case,

$$r^e_{t,t+1} = \Big[\{(1-\tau_{1,t+1})(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1}) + 1\}\, k_{1m,t+1}$$

$$+ (1-\tau_{1,t+1})(r_{1u,t+1} - \delta_{1u} + 1)k_{1u,t+1}\Big]$$

$$/(k_{1m,t+1} + (1-\tau_{1,t})k_{1u,t+1}) - 1$$

- After-tax NIPA earnings/value (when $\phi = \phi' = 0$):

$$\frac{e_{t+1}}{v_t} = \frac{(1-\tau_{1,t+1})\left[(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1})k_{1m,t+1} + r_{1u,t+1}k_{1u,t+1} - x_{1u,t+1}\right]}{k_{1m,t+1} + (1-\tau_{1,t})k_{1u,t+1}}$$

$$= \Big[\{(1-\tau_{1,t+1})(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1}) + 1\}\, k_{1m,t+1}$$

$$+ (1-\tau_{1,t+1})(r_{1u,t+1}k_{1u,t+1} - x_{1u,t+1}) + (1-\tau_{1,t})k_{1u,t+1}\Big]$$

$$/(k_{1m,t+1} + (1-\tau_{1,t})k_{1u,t+1}) - 1$$

$$= \Big[\{(1-\tau_{1,t+1})(r_{1m,t+1} - \delta_{1m} - \tau_{1k,t+1}) + 1\}\, k_{1m,t+1}$$

$$+ (1-\tau_{1,t+1})(r_{1u,t+1} - \delta_{1u} + 1)k_{1u,t+1}$$

$$- (1-\tau_{1,t+1})(1+\eta)k_{1u,t+2} + (1-\tau_{1,t})k_{1u,t+1}\Big]$$

$$/(k_{1m,t+1} + (1-\tau_{1,t})k_{1u,t+1}) - 1$$

Notice that this is not exactly equal to the return on capital or on corporate equities because of the intangible capital.