Computing Asset Returns in a Standard Growth Model

1. The Optimization Problems

The household's problem: choose sequences of consumptions, investments, and hours of leisure/work that solve

$$\begin{aligned}
&\max \; E[\sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \,|\, X_0] \\
&\text{subject to} \;\; (1+\tau_{c,t})c_t + x_{1,t} + x_{2,t} = (1-\tau_{1,t})r_{1,t}k_{1,t} + (1-\tau_{2,t})r_{2,t}k_{2,t} \\
&\qquad\qquad + (1-\tau_{n,t})w_t n_t + \tau_{1,t}\delta_1 k_{1,t} + \tau_{2,t}\delta_2 k_{2,t} + T_t \\
&\qquad\qquad n_t + l_t = 1 \\
&\qquad\qquad k_{i,t+1} = (1-\delta_i)k_{i,t} + x_{i,t}, \quad i = 1,2
\end{aligned}$$

where τ's are stochastic.

Problem for a firm in sector i: choose capital and labor to solve

$$\begin{aligned}
&\max \;\; p_i y_i - r_i k_i - w n_i \\
&\qquad \text{subject to} \;\; y_i = k_i^{\theta_i}(z n_i)^{1-\theta_i}
\end{aligned}$$

where $\log z$ is a unit-root process and θ_i is also stochastic. We will call Sector 1 the corporate sector and Sector 2 the noncorporate sector.

The final goods firm's problem: choose intermediate and final outputs to solve

$$\begin{aligned}
&\max \;\; y - p_1 y_1 - p_2 y_2 \\
&\qquad \text{subject to} \;\; y = (a_1 y_1^{\rho} + a_2 y_2^{\rho})^{\frac{1}{\rho}}.
\end{aligned}$$

The government budget constraint is:

$$g_t + T_t = \tau_{c,t}c_t + \tau_{1,t}r_{1,t}k_{1,t} + \tau_{2,t}r_{2,t}k_{2,t} + \tau_{n,t}w_t n_t - \tau_{1,t}\delta_1 k_{1,t} - \tau_{2,t}\delta_2 k_{2,t}$$

where g is stochastic.

Additionally, in equilibrium, we need:

$$n_t = n_{1,t} + n_{2,t}$$

$$y_t = c_t + x_{1,t} + x_{2,t} + g_t.$$

2. The First-order Conditions

The household: has first-order conditions:

$$\frac{U_c(c_t,l_t)}{1+\tau_{c,t}} = \beta E\left[\frac{U_c(c_{t+1},l_{t+1})}{1+\tau_{c,t+1}}\left((1-\tau_{i,t+1})(r_{i,t+1}-\delta_i)+1\right) \mid k_{1,t},k_{2,t},s_t\right], \quad i=1,2$$

$$U_l(c_t,l_t) = U_c(c_t,l_t)w_t(1-\tau_{n,t})/(1+\tau_{c,t})$$

where $s_t = [\eta_t, \tau_{c,t}, \tau_{1,t}, \tau_{2,t}, \tau_{n,t}, g_t, \theta_{1,t}, \theta_{2,t}]'$ and η_t is the innovation to technology,

$$\log\ z_{t+1} = \log\ z_t + \log(1+\gamma) + \eta_{t+1}.$$

The firms: have first-order conditions:

$$r_i = p_i\theta_i k_i^{\theta_i - 1}(zn_i)^{1-\theta_i}, \quad i=1,2$$

$$w = p_i(1-\theta_i)k_i^{\theta_i}n_i^{-\theta_i}z^{1-\theta_i}, \quad i=1,2$$

$$p_i = a_i(y/y_i)^{1-\rho}, \quad i=1,2$$

3. Equilibrium Decision Functions

Suppose that we have computed functions for investment and hours of work in the corporate sector, namely $x_{1,t} = x_1(k_{1,t}, k_{2,t}, s_t)$ and $n_{1,t} = n_1(k_{1,t}, k_{2,t}, s_t)$. Claim: we can back out everything else using static first-order conditions and values for the states $k_{1,t}$, $k_{2,t}$, and s_t.

- Step 1. Set pre-tax wages paid in the two sectors:

$$p_1(1-\theta_1)k_1^{\theta_1}n_1^{-\theta_1}z^{1-\theta_1} = p_2(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}$$

  which implies a price ratio of

$$\frac{p_1}{p_2} = \frac{(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}}{(1-\theta_1)k_1^{\theta_1}n_1^{-\theta_1}z^{1-\theta_1}}.$$

- Step 2. Equate price ratio from Step 1 to price ratio found from final goods producer's problem:

$$\frac{(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}}{(1-\theta_1)k_1^{\theta_1}n_1^{-\theta_1}z^{1-\theta_1}} = \frac{a_1 y_1^{\rho-1}}{a_2 y_2^{\rho-1}} = \frac{a_1(k_1^{\theta_1}(zn_1)^{1-\theta_1})^{\rho-1}}{a_2(k_2^{\theta_2}(zn_2)^{1-\theta_2})^{\rho-1}}.$$

  This gives us an expression for n_2:

$$n_2 = \left[\left(\frac{1-\theta_2}{1-\theta_1}\right)\left(\frac{a_2}{a_1}\right)\left(\frac{k_2^{\theta_2\rho}z^{-\theta_2\rho}}{k_1^{\theta_1\rho}z^{-\theta_1\rho}}\right)n_1^{1-\rho(1-\theta_1)}\right]^{\frac{1}{1-\rho(1-\theta_2)}}.$$

- Step 3. With the states $k_{1,t}$, $k_{2,t}$, and s_t and the hours $n_{1,t}$ and $n_{2,t}$, we can get output levels, prices, and leisure:

$$\begin{aligned}
y_{i,t} &= k_{i,t}^{\theta_{i,t}}(z_t n_{i,t})^{1-\theta_{i,t}}, \quad i = 1,2 \\
y_t &= (a_1 y_{1,t}^{\rho} + a_2 y_{2,t}^{\rho})^{\frac{1}{\rho}} \\
p_{i,t} &= a_i(y_t/y_{i,t})^{1-\rho}, \quad i = 1,2 \\
r_{i,t} &= p_{i,t}\theta_{i,t}y_{i,t}/k_{i,t}, \quad i = 1,2 \\
w_t &= p_{1,t}(1-\theta_{1,t})y_{1,t}/n_{1,t} \\
l_t &= 1 - n_{2,t} - n_{1,t}.
\end{aligned}$$

- Step 4. With leisure l_t, the wage rate w_t, and the tax on hours $\tau_{n,t}$, we can back out consumption from

$$U_c(c_t, l_t) = U_c(c_t, l_t) w_t (1 - \tau_{n,t})/(1 + \tau_{c,t}).$$

- Step 5. With corporate investment $x_{1,t}$, consumption c_t, total output y_t, and government spending g_t, we can back out total investment and investment in noncorporate capital:

$$x_t = y_t - c_t - g_t$$

$$x_{2,t} = x_t - x_{1,t}$$

- Step 6. With the capital stocks and investments, we can determine next period's capital stocks:

$$k_{i,t+1} = (1 - \delta_i) k_{i,t} + x_{i,t}, \qquad i = 1, 2.$$

We now need to compute $x_{1,t}$ and $n_{1,t}$. Since we will focus on business-cycle sized shocks, the approximation method we use to compute these functions assumes that they are close to log-linear functions of the state.

Because z is nonstationary, we first transform our nonstationary equations to stationary ones. We will use a $\tilde{}$ to denote transformed variables. Thus, $\tilde{x}_t = x_t/z_t$.

For our computation, we work with the preferences

$$U(c, l) = \frac{(c\, l^{\psi})^{1-\sigma}}{1 - \sigma}.$$

When we write this as a function of transformed variables we replace c with $z\tilde{c}$. Therefore, our first order conditions for the household can be written as follows:

$$1 = \beta E_t \Bigg\{ ((1+\gamma) e^{\eta_{t+1}})^{-\sigma} \frac{\tilde{c}_{t+1}^{-\sigma} l_{t+1}^{\psi(1-\sigma)} (1 + \tau_{c,t})}{\tilde{c}_t^{-\sigma} l_t^{\psi(1-\sigma)} (1 + \tau_{c,t+1})} \Big[(1 - \tau_{i,t+1})(p_{i,t+1} \theta_{i,t+1} \tilde{k}_{i,t+1}^{\theta_{i,t+1}-1} n_{i,t+1}^{1-\theta_{i,t+1}} - \delta_i) + 1 \Big] \Bigg\}, \quad i = 1, 2 \qquad (3.1)$$

and similarly for the second dynamic equation. The equation for the wage also relies on the specification for U. When transformed this equation is given by

$$\psi \tilde{c}_t / l_t = (1-\tau_{n,t}) p_{1,t} (1-\theta_{1,t}) \tilde{k}_{1,t}^{\theta_{1,t}} n_{1,t}^{-\theta_{1,t}} / (1+\tau_{c,t}).$$

Notice that z_t does not appear in this equation.

The equations for capital accumulation change slightly when we rewrite the equations in terms of stationary variables. These are now given by

$$\tilde{k}_{i,t+1} = \left[(1-\delta_i)\tilde{k}_{i,t} + \tilde{x}_{i,t}\right] e^{-\eta_{t+1}}/(1+\gamma), \qquad i = 1, 2. \tag{3.2}$$

We turn next to solving our system of difference equations. We use the static first-order conditions to eliminate n_{2t}, $\tilde{y}_{1,t}$, $\tilde{y}_{2,t}$, $\tilde{y}_t$, $p_{1,t}$, $p_{2,t}$, $r_{1,t}$, $r_{2,t}$, $\tilde{w}_t$, l_t, $\tilde{c}_t$, $\tilde{x}_t$, and $\tilde{x}_{2,t}$ in (3.1) and (3.2). We then linearize these equations and get a system of equations that look like

$$E_t \left\{ \Psi_1 \begin{bmatrix} \hat{k}_{1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Psi_2 \begin{bmatrix} \hat{n}_{1,t+1} \\ \hat{x}_{1,t+1} \end{bmatrix} + \Psi_3 \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Psi_4 \begin{bmatrix} \hat{n}_{1,t} \\ \hat{x}_{1,t} \end{bmatrix} + \Psi_5 \hat{s}_{t+1} + \Psi_6 \hat{s}_t \right\} = 0$$

$$\Phi_1 \begin{bmatrix} \hat{k}_{1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Phi_2 \begin{bmatrix} \hat{n}_{1,t+1} \\ \hat{x}_{1,t+1} \end{bmatrix} + \Phi_3 \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Phi_4 \begin{bmatrix} \hat{n}_{1,t} \\ \hat{x}_{1,t} \end{bmatrix} + \Phi_5 \hat{s}_{t+1} + \Phi_6 \hat{s}_t = 0$$

where $\hat{}$ denotes stationary variables that have been logged and demeaned, e.g., $\hat{k}_{1,t+1} = \log \tilde{k}_{1,t+1} - \log k_1$. All variables in $\hat{s}_t$ are in levels – with means subtracted – except government spending which is logged and demeaned. We assume that the process for $\hat{s}_t$ is given by

$$\hat{s}_{t+1} = P\hat{s}_t + \epsilon_{t+1}$$

where ϵ is a mean 0, serially uncorrelated vector of shocks.

We are looking for a solution of the form:

$$\begin{bmatrix} \hat{n}_{1,t} \\ \hat{x}_{1,t} \end{bmatrix} = A \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + B\hat{s}_t.$$

Since the system of equations is linear, we can ignore the stochastic terms while solving A. This matrix is found by computing the eigenvalues and eigenvectors of the following matrix:

$$-\begin{bmatrix} \Phi_1 & 0 \\ \Psi_1 & \Psi_2 \end{bmatrix}^{-1} \begin{bmatrix} \Phi_3 & \Phi_4 \\ \Psi_3 & \Psi_4 \end{bmatrix}$$

(or generalized eigenvalues if Ψ_2 is not invertible). Notice that a 0 is written in for Φ_2 since the capital accumulation equations don't depend on decision variables in $t+1$. Suppose this matrix is equal to $V\Lambda V^{-1}$ where V are eigenvectors and Λ are eigenvalues. Assuming the eigenvalues are ordered so that the first two are inside the unit circle, we have our solution:

$$A = V_{21}V_{11}^{-1}$$

where V_{11} and V_{21} are 2×2 partitions of V (i.e., the upper left and lower left partitions, respectively).

Given A, we can compute B using the method of undetermined coefficients. First note that

$$\begin{bmatrix} \hat{k}_{1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} = -\Phi_1^{-1}(\Phi_3 + \Phi_4 A) \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} - \Phi_1^{-1}\Phi_5 \hat{s}_{t+1} - \Phi_1^{-1}(\Phi_6 + \Phi_4 B)\hat{s}_t.$$

Next note that B must be set so that

$$\begin{aligned}
0 = E_t\Bigg\{ & \Psi_1 \begin{bmatrix} \hat{k}_{1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Psi_2 \begin{bmatrix} \hat{n}_{1,t+1} \\ \hat{x}_{1,t+1} \end{bmatrix} + \Psi_3 \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Psi_4 \begin{bmatrix} \hat{n}_{1,t} \\ \hat{x}_{1,t} \end{bmatrix} + \Psi_5 \hat{s}_{t+1} + \Psi_6 \hat{s}_t \Bigg\} \\
= E_t\Bigg\{ & \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_3 + \Phi_4 A) + \Psi_3 + \Psi_4 A\right] \left[\, \hat{k}_{1,t} \quad \hat{k}_{2,t} \,\right]' \\
& + \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 + \Psi_5 + \Psi_2 B\right] \hat{s}_{t+1} \\
& \quad + \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B\right] \hat{s}_t \Bigg\} \\
= \; & \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_3 + \Phi_4 A) + \Psi_3 + \Psi_4 A\right] \left[\, \hat{k}_{1,t} \quad \hat{k}_{2,t} \,\right]' \\
& + \left[-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P + \Psi_5 P + \Psi_2 B P \right. \\
& \quad \left. - (\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B\right] \hat{s}_t
\end{aligned}$$

is satisfied. For this to be satisfied, we need the coefficient on $\hat{s}_t$ equal to zero. Thus, we have 28 equations in 28 unknowns (i.e., the elements of B):

$$
\begin{aligned}
0_{4\times 7} = &-(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P + \Psi_5 P + \Psi_2 B P \\
&- (\Psi_1 + \Psi_2 A)\Phi_1^{-1}(\Phi_6 + \Phi_4 B) + \Psi_6 + \Psi_4 B.
\end{aligned}
$$

This can be written as a simple linear system in $vec(B)$ if we apply the standard formula: $vec(ABC) = [C' \otimes A]vec(B)$. Doing this we get:

$$
\begin{aligned}
vec(B) = &\left\{[P' \otimes \Psi_2] + [I \otimes -(\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_4] + [I \otimes \Psi_4]\right\}^{-1} \\
&vec((\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_5 P - \Psi_5 P + (\Psi_1 + \Psi_2 A)\Phi_1^{-1}\Phi_6 - \Psi_6)
\end{aligned}
$$

4. Steady State Equations

For our linear approximation, we need to compute the steady state of the transformed economy. The nonlinear equations to be solved are:

$$1 = \beta(1+\gamma)^{-\sigma}\left[(1-\tau_i)(p_i\theta_i(n_i/\tilde{k}_i)^{1-\theta_i} - \delta_i) + 1\right], \quad i = 1, 2 \tag{4.1}$$

$$\psi\tilde{c} = (1-\tau_n)p_1(1-\theta_1)(\tilde{k}_1/n_1)^{\theta_1}(1 - n_1 - n_2)/(1+\tau_c) \tag{4.2}$$

$$\tilde{k}_i = [(1-\delta_1)\tilde{k}_i + \tilde{x}_i]/(1+\gamma), \quad i = 1, 2 \tag{4.3}$$

$$\tilde{y} = \tilde{c} + \tilde{x}_1 + \tilde{x}_2 + \tilde{g} \tag{4.4}$$

$$\tilde{y} = (a_1\tilde{y}_1^{\rho} + a_2\tilde{y}_2^{\rho})^{\frac{1}{\rho}} \tag{4.5}$$

$$\tilde{y}_i = (n_i/\tilde{k}_i)^{1-\theta_i}\tilde{k}_i, \quad i = 1, 2 \tag{4.6}$$

$$p_i = a_1(\tilde{y}/\tilde{y}_i)^{1-\rho}, \quad i = 1, 2 \tag{4.7}$$

$$p_1(1-\theta_1)(\tilde{k}_1/n_1)^{\theta_1} = p_2(1-\theta_2)(\tilde{k}_2/n_2)^{\theta_2} \tag{4.8}$$

which is 12 equations in 12 unknowns: $\tilde{k}_1$, n_1, p_1, $\tilde{k}_2$, n_2, p_2, $\tilde{c}$, $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{y}$, $\tilde{y}_1$, $\tilde{y}_2$.

Start with a guess for $\tilde{y}_1/\tilde{y}$. With $\tilde{y}_1/\tilde{y}$, we get $\tilde{y}_2/\tilde{y}$ from final goods technology (4.5). With the output ratios, we can get p_1 and p_2 from (4.7). The prices then are used to get $n_1/\tilde{k}_1$ and $n_2/\tilde{k}_2$ from (4.1). We can use these results to check if (4.8) is satisfied.

Now make a guess for n_1. Given a guess for n_1 and our derived ratio, $n_1/\tilde{k}_1$ we get $\tilde{k}_1$. This gives us $\tilde{y}_1$ from (4.6) and $\tilde{y}$ if we divide $\tilde{y}_1$ by our guess for $\tilde{y}_1/\tilde{y}$. We then get $\tilde{y}_2$ residually from (4.5). With $\tilde{y}_2$ and $\tilde{k}_2/n_2$, we get $\tilde{k}_2$ from (4.6) and n_2 from the noncorporate production function given values for $\tilde{y}_2$ and $\tilde{k}_2$. With $\tilde{k}_1$ and $\tilde{k}_2$ we get $\tilde{x}_1$ and $\tilde{x}_2$ from (4.3). From (4.4) we get $\tilde{c}$ since we know $\tilde{y}$, $\tilde{x}_1$ and $\tilde{x}_2$. We can then check to make sure that (4.2) is satisfied.

5. Test Cases

We compare the results to those computed in "A Progress Report on Business Cycle Models" (QR, Fall 1994) where technology is assumed to be an AR(1) process. To make things comparable we set the serial correlation coefficient near 1 in the older programs. We set $a_1 = a_2$ and $\rho = 1$ in the new program. These are one-sector versions of the model.

6. Computing Returns

We will follow the approach in Jermann (JME, 1998) who uses a log-linear approximation (as done above) to compute the law of motion for the capital stocks and shocks and nonlinear asset-pricing formulas to compute returns. When he compares his results to those found with nonlinear solution methods, he finds equity premia that are very close.

Define the vector X_t as $X_t = [\hat{k}_{1,t}, \hat{k}_{2,t}, \hat{s}_t]$. From our earlier algebra, we have

$$X_{t+1} = A_o X_t + B_o \epsilon_{t+1}$$

where we will assume that ϵ is a multivariate normal i.i.d. shock. Let λ_t be the logarithm of the marginal utility – after transforming variables – with the mean subtracted off. The expression for λ_t is found as follows

$$
\begin{aligned}
\lambda_t &= \ln\left(\tilde{c}_t^{-\sigma} l_t^{\psi(1-\sigma)}\right) - \ln\left(\tilde{c}^{-\sigma} l^{\psi(1-\sigma)}\right) \\
&= -\sigma \hat{c}_t + \psi(1-\sigma)\hat{l}_t \\
&\simeq \alpha_1 \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \alpha_2 \begin{bmatrix} \hat{n}_{1,t} \\ \hat{x}_{1,t} \end{bmatrix} + \alpha_3 \hat{s}_t \\
&= (\alpha_1 + \alpha_2 A) \begin{bmatrix} \hat{k}_{1,t} \\ \hat{k}_{2,t} \end{bmatrix} + (\alpha_2 B + \alpha_3)\hat{s}_t \\
&= [\alpha_1 + \alpha_2 A, \alpha_2 B + \alpha_3]' X_t \\
&\equiv \alpha' X_t.
\end{aligned}
$$

We want to use this formula to compute an analytical expression for the expected gross risk-free rate, $ER_{f,t}$, where $R_{f,t}$ is given by:

$$
\begin{aligned}
R_{f,t} &= \left[\beta E_t \frac{U_c(c_{t+1}, l_{t+1})}{U_c(c_t, l_t)}\right]^{-1} \\
&= \left[\beta E_t \frac{c_{t+1}^{-\sigma} l_{t+1}^{\psi(1-\sigma)}}{c_t^{-\sigma} l_t^{\psi(1-\sigma)}}\right]^{-1} \\
&= \left[\beta(1+\gamma)^{-\sigma} E_t e^{-\sigma\eta_{t+1} + \lambda_{t+1} - \lambda_t}\right]^{-1}
\end{aligned}
$$

Since ϵ is normal, X is normal and λ is normal. Using the fact that $Ee^x = e^{1+\gamma+\sigma^2/2}$ when x is normally distributed with mean $1+\gamma$ and variance σ^2, we have

$$
\begin{aligned}
ER_t^f &= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[E_t \exp(-\sigma\eta_{t+1} + \lambda_{t+1} - \lambda_t)\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[E_t \exp(-\sigma\rho_\eta\eta_t - \sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - \lambda_t + E_t\lambda_{t+1} - E_t\lambda_{t+1})\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\eta\eta_t + E_t\lambda_{t+1} - \lambda_t) E_t \exp(-\sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - E_t\lambda_{t+1})\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\eta\eta_t + E_t\lambda_{t+1} - \lambda_t) \exp(\frac{1}{2}\mathrm{var}_t[-\sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}])\right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\left[\exp(-\sigma\rho_\eta\eta_t + E_t\lambda_{t+1} - \lambda_t)\exp(\frac{1}{2}\text{var}[-\sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}])\right]^{-1} \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} E\exp(\sigma\rho_\eta\eta_t - E_t\lambda_{t+1} + \lambda_t)\exp(-\frac{1}{2}\text{var}[-\sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}]) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\text{var}[\sigma\rho_\eta\eta_t - E_t\lambda_{t+1} + \lambda_t] - \frac{1}{2}\text{var}[-\sigma\epsilon_{\eta,t+1} + \lambda_{t+1} - E_t\lambda_{t+1}]\right) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\text{var}([\sigma\rho_\eta e' + \alpha'(I-A_o)]X_t) - \frac{1}{2}\text{var}([-\sigma e' + \alpha']B_o\epsilon_{t+1})\right) \\
&= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\mu' V_x \mu - \frac{1}{2}\kappa' B_o B_o' \kappa\right)
\end{aligned}
$$

where e is a 9×1 vector of zeros except for $e(3) = 1$ and

$$
\begin{aligned}
\mu &= [\sigma\rho_\eta e' + \alpha'(I - A_o)]' \\
\kappa &= [-\sigma e' + \alpha']' \\
V_x &= A_o V_z A_o' + B_o B_o'.
\end{aligned}
$$

To compute the expected return on corporate equities, we need to simulate the model and compute

$$
r^e_{t,t+1} = \frac{V_{t+1}(X_{t+1}) + d_{t+1}(X_{t+1})}{V_t(X_t)}.
$$

Therefore, we want to compute the following objects:

- Corporate earnings:

$$
e_{1,t} = p_{1,t}y_{1,t} - w_t n_{1,t} - \delta_1 k_{1,t} = (r_{1,t} - \delta_1)k_{1,t}.
$$

- After-tax corporate earnings:

$$
(1 - \tau_{1,t})e_{1,t} = (1 - \tau_{1,t})(r_{1,t} - \delta_1)k_{1,t}.
$$

- Dividends (=After-tax corporate earnings less retained earnings):

$$
d_t = (1 - \tau_{1,t})e_{1,t} - k_{1,t+1} + k_{1,t}.
$$

- The ex-dividend value of the corporations:

$$V_t = k_{1,t+1}$$

(NOTE: the pre-dividend value is $[(1-\tau_{1,t})(r_{1,t}-\delta)+1]k_{1,t}$.)

- The post-dividend return on corporate equities

$$\begin{aligned} r^e_{t,t+1} &= \frac{d_{t+1}+V_{t+1}}{V_t} - 1 \\ &= \frac{(1-\tau_{1,t+1})(r_{1,t+1}-\delta_1)k_{1,t+1} - k_{1,t+2} + k_{1,t+1} + k_{1,t+2}}{k_{1,t+1}} - 1 \\ &= (1-\tau_{1,t+1})(r_{1,t+1}-\delta_1). \end{aligned}$$

7. Variations

7.1. Adjustment Costs on Investment

Suppose that the capital accumulation equation is given by

$$k_{i,t+1} = (1-\delta_i)k_{i,t} + x_{i,t} - \phi\left(\frac{x_{i,t}}{k_{i,t}}\right)k_{i,t}, \qquad i = 1,2.$$

The case we studied earlier had $\phi = 0$. Now consider $\phi(x/k) > 0$.

The intertemporal first-order condition for the household's problem becomes:

$$\begin{aligned} \frac{U_c(c_t,l_t)}{(1+\tau_{c,t})(1-\phi'_{i,t})} = \beta E\Bigg[\frac{U_c(c_{t+1},l_{t+1})}{(1+\tau_{c,t+1})}\Bigg\{& r_{i,t+1} - \tau_{i,t+1}(r_{i,t+1}-\delta_i) \\ &+ \frac{1}{1-\phi'_{i,t+1}}\left[1-\delta_i-\phi_{i,t+1}+\phi'_{i,t+1}\frac{x_{i,t+1}}{k_{i,t+1}}\right]\Bigg\}\ \Bigg|\ k_{1,t},k_{2,t},s_t\Bigg] \end{aligned}$$

where $\phi_{i,t}$ is shorthand notation for $\phi(x_{i,t}/k_{i,t})$ and $\phi'_{i,t}$ is the first derivative of this function evaluated at $x_{i,t}/k_{i,t}$.

We will use the following functional form

$$\phi_{i,t} = \frac{b_i}{2}\left(\frac{x_{i,t}}{k_{i,t}} - \gamma - \delta_i\right)^2.$$

In this case, when we normalize the capital stocks and investments, the capital accumulation equation becomes

$$\tilde{k}_{i,t+1} = \left[(1-\delta_i)\tilde{k}_{i,t} + \tilde{x}_{i,t} - \frac{b_i}{2}\left(\frac{\tilde{x}_{i,t}}{\tilde{k}_{i,t}} - \gamma - \delta_i\right)^2 \tilde{k}_{i,t}\right] e^{-\eta_{t+1}}/(1+\gamma).$$

Note that the steady state does not change but the first-order conditions do.

7.2. No Unit Root in Technology

Suppose we assume that

$$\log\, z_{t+1} = \rho_z z_t + \eta_{t+1}$$

and

$$y_{i,t} = k_i^{\theta_{i,t}} (z_t (1+\gamma)^t n_{i,t})^{1-\theta_{i,t}}$$

for $i = 1, 2$. Then, assuming that $\tilde{x}_t = x_t/(1+\gamma)^t$, we have

$$\begin{aligned} 1 = \beta(1+\gamma)^{-\sigma} E_t \Bigg\{ & \frac{\tilde{c}_{t+1}^{-\sigma} l_{t+1}^{\psi(1-\sigma)}(1+\tau_{c,t})}{\tilde{c}_t^{-\sigma} l_t^{\psi(1-\sigma)}(1+\tau_{c,t+1})} \\ & \left[(1-\tau_{i,t+1})(p_{i,t+1}\theta_{i,t+1}\tilde{k}_{i,t+1}^{\theta_{i,t+1}-1}(z_{t+1}n_{i,t+1})^{1-\theta_{i,t+1}} - \delta_i) + 1\right] \Bigg\} \end{aligned}$$

$$\psi \tilde{c}_t / l_t = (1-\tau_{n,t}) p_{1,t} (1-\theta_{1,t}) (\tilde{k}_{1,t}/n_{1,t})^{\theta_{1,t}} z_t^{1-\theta_{1,t}} / (1+\tau_{c,t})$$

$$\tilde{k}_{i,t+1} = \left[(1-\delta_i)\tilde{k}_{i,t} + \tilde{x}_{i,t}\right]/(1+\gamma)$$

For this case, the expected risk-free rate is given by

$$\begin{aligned} ER_t^f &= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\text{var}[-E_t\lambda_{t+1} + \lambda_t] - \frac{1}{2}\text{var}[\lambda_{t+1} - E_t\lambda_{t+1}]\right) \\ &= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\text{var}(\alpha'(I-A_o)X_t) - \frac{1}{2}\text{var}(\alpha' B_o \epsilon_{t+1})\right) \\ &= \frac{1}{\beta(1+\gamma)^{-\sigma}} \exp\left(\frac{1}{2}\alpha'(I-A_o)V_x(I-A_o)'\alpha - \frac{1}{2}\alpha' B_o B_o' \alpha\right). \end{aligned}$$

7.3. Add Unmeasured Capital, Population Growth, and Property Taxes

The household's new problem: choose sequences of consumptions, investments, and hours of leisure/work that solve

$$\max \; E[\sum_{t=0}^{\infty} \beta^t \, N_t U(c_t, l_t) \,|\, X_0]$$

$$\begin{aligned}
\text{subject to} \;\; & N_t \Big[(1+\tau_{c,t})c_t + x_{m1,t} + x_{u1,t} + x_{2,t}\Big] \\
&= N_t \Big[r_{m1,t}k_{m1,t} + r_{u1,t}k_{u1,t} + r_{2,t}k_{2,t} + w_t n_t \\
&\quad - \tau_{1,t}\left[(r_{m1,t} - \delta_{m1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t} - \tau_{k1,t}k_{m1,t}\right] \\
&\quad - \tau_{2,t}\left[(r_{2,t} - \delta_2)k_{2,t} - \tau_{k2,t}k_{2,t}\right] - \tau_{n,t}w_t n_t \\
&\quad - \tau_{k1,t}k_{m1,t} - \tau_{k2,t}k_{2,t} + T_t \Big] \\
& N_{t+1} = (1+\eta)N_t \\
& n_t + l_t = 1 \\
& (1+\eta)k_{i,t+1} = (1-\delta_i)k_{i,t} + x_{i,t}, \quad i = m1, u1, 2
\end{aligned}$$

where N_t is the size of the household at date t and all variables are in per-capita terms. Therefore, n_t is the fraction of family members working and l_t is the fraction enjoying leisure.

Problem for a firm in sector 1: choose capital and labor to solve

$$\begin{aligned}
\max \;\; & p_1 y_1 - r_{m1}k_{m1} - r_{u1}k_{u1} - w n_1 \\
\text{subject to} \;\; & y_1 = k_{m1}^{\theta_{m1}} k_{u1}^{\theta_{u1}} (z n_1)^{1-\theta_{m1}-\theta_{u1}}
\end{aligned}$$

while the problem in sector 2 is the same as before.

The final goods firm's problem: same as before.

The government budget constraint:

$$\begin{aligned}
g_t + T_t = \tau_{c,t}c_t + \tau_{1,t}&\left[(r_{m1,t} - \delta_{m1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t} - \tau_{k1,t}k_{m1,t}\right] \\
&+ \tau_{2,t}\left[(r_{2,t} - \delta_2)k_{2,t} - \tau_{k2,t}k_{2,t}\right] + \tau_{n,t}w_t n_t + \tau_{k1,t}k_{m1,t} - \tau_{k2,t}k_{2,t}
\end{aligned}$$

Additionally, in equilibrium, we need:

$$n_t = n_{1,t} + n_{2,t}$$

$$y_t = c_t + x_{m1,t} + x_{u1,t} + x_{2,t} + g_t.$$

7.3.1. The new first-order conditions

Everything is as before except now we have:

$$\begin{aligned}
r_{m1} &= p_1 \theta_{m1} k_{m1}^{\theta_{m1}-1} k_{u1}^{\theta_{u1}} (zn_1)^{1-\theta_{m1}-\theta_{u1}} \\
r_{u1} &= p_1 \theta_{u1} k_{m1}^{\theta_{m1}} k_{u1}^{\theta_{u1}-1} (zn_1)^{1-\theta_{m1}-\theta_{u1}} \\
w &= p_1 (1-\theta_{m1}-\theta_{u1}) k_{m1}^{\theta_{m1}} k_{u1}^{\theta_{u1}} n_1^{-\theta_{m1}-\theta_{u1}} z^{1-\theta_{m1}-\theta_{u1}} \\
w &= p_2 (1-\theta_2) k_2^{\theta_2} n_2^{-\theta_2} z^{1-\theta_2} \\
k_{m1}' &= [(1-\delta_{m1}) k_{m1} + x_{m1}]/(1+\eta) \\
k_{u1}' &= [(1-\delta_{u1}) k_{u1} + x_{u1}]/(1+\eta) \\
k_2' &= [(1-\delta_2) k_2 + x_2]/(1+\eta)
\end{aligned}$$

and the dynamic first order conditions for the household are

$$\begin{aligned}
\frac{U_c(c_t, l_t)}{1+\tau_{c,t}} &= \beta E \left[\frac{U_c(c_{t+1}, l_{t+1})}{1+\tau_{c,t+1}} \left((1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1}) + 1 \right) \mid X_t \right] \\
\frac{U_c(c_t, l_t)}{1+\tau_{c,t}} &= \beta E \left[\frac{U_c(c_{t+1}, l_{t+1})}{1+\tau_{c,t+1}} (1-\tau_{1,t+1}) \left(r_{u1,t+1} - \delta_{u1} + 1 \right) / (1-\tau_{1,t}) \mid X_t \right] \\
\frac{U_c(c_t, l_t)}{1+\tau_{c,t}} &= \beta E \left[\frac{U_c(c_{t+1}, l_{t+1})}{1+\tau_{c,t+1}} \left((1-\tau_{2,t+1})(r_{2,t+1} - \delta_2 - \tau_{k2,t+1}) + 1 \right) \mid X_t \right]
\end{aligned}$$

where $X_t = [k_{m1,t}, k_{u1,t}, k_{2,t}, s_t]'$ and $s_t = [\eta_t, \tau_{c,t}, \tau_{1,t}, \tau_{2,t}, \tau_{n,t}, g_t, \theta_{1,t}, \theta_{2,t}, \tau_{p1,t}, \tau_{p2,t}]'$ and now includes property tax rates.

7.3.2. New equilibrium decision functions

In this case, we have three dynamic equations and will therefore compute three decision functions, namely $x_{m1,t} = x_{m1}(k_{m1,t}, k_{u1,t}, k_{2,t}, s_t)$, $x_{u1,t} = x_{u1}(k_{m1,t}, k_{u1,t}, k_{2,t}, s_t)$ and $n_{1,t} = n_1(k_{1,t}, k_{2,t}, s_t)$. With these, we can back out everything else as before.

- Step 1. Set pre-tax wages paid in the two sectors:

$$p_1(1-\theta_{m1}-\theta_{m2})k_{m1}^{\theta_{m1}}k_{u1}^{\theta_{u1}}n_1^{-\theta_{m1}-\theta_{u1}}z^{1-\theta_{m1}-\theta_{u1}} = p_2(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}$$

  which implies a price ratio of

$$\frac{p_1}{p_2} = \frac{(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}}{(1-\theta_{m1}-\theta_{m2})k_{m1}^{\theta_{m1}}k_{u1}^{\theta_{u1}}n_1^{-\theta_{m1}-\theta_{u1}}z^{1-\theta_{m1}-\theta_{u1}}}.$$

- Step 2. Equate price ratio from Step 1 to price ratio found from final goods producer's problem:

$$\frac{(1-\theta_2)k_2^{\theta_2}n_2^{-\theta_2}z^{1-\theta_2}}{(1-\theta_{m1}-\theta_{m2})k_{m1}^{\theta_{m1}}k_{u1}^{\theta_{u1}}n_1^{-\theta_{m1}-\theta_{u1}}z^{1-\theta_{m1}-\theta_{u1}}}. = \frac{a_1y_1^{\rho-1}}{a_2y_2^{\rho-1}} = \frac{a_1(k_1^{\theta_1}(zn_1)^{1-\theta_1})^{\rho-1}}{a_2(k_2^{\theta_2}(zn_2)^{1-\theta_2})^{\rho-1}}.$$

  This gives us an expression for n_2:

$$n_2 = \left[\left(\frac{1-\theta_2}{1-\theta_{m1}-\theta_{u1}}\right)\left(\frac{a_2}{a_1}\right)\left(\frac{k_2^{\theta_2\rho}z^{-\theta_2\rho}}{k_{m1}^{\theta_{m1}\rho}k_{u1}^{\theta_{u1}\rho}z^{-\theta_1\rho}}\right)n_1^{1-\rho(1-\theta_{m1}-\theta_{u1})}\right]^{\frac{1}{1-\rho(1-\theta_2)}}.$$

- Step 3. With the states $k_{m1,t}$, $k_{u1,t}$, $k_{2,t}$, and s_t and the hours $n_{1,t}$ and $n_{2,t}$, we can get output levels, prices, and leisure:

$$\begin{aligned}
y_{1,t} &= k_{m1,t}^{\theta_{m1,t}}k_{u1,t}^{\theta_{u1,t}}(z_tn_{i,t})^{1-\theta_{m1,t}-\theta_{m2,t}}\\
y_{i,t} &= k_{2,t}^{\theta_{2,t}}(z_tn_{2,t})^{1-\theta_{2,t}}\\
y_t &= (a_1y_{1,t}^{\rho}+a_2y_{2,t}^{\rho})^{\frac{1}{\rho}}\\
p_{i,t} &= a_i(y_t/y_{i,t})^{1-\rho}, \quad i=1,2\\
r_{m1,t} &= p_{1,t}\theta_{m1,t}y_{1,t}/k_{m1,t}\\
r_{u1,t} &= p_{1,t}\theta_{u1,t}y_{1,t}/k_{u1,t}\\
r_{2,t} &= p_{2,t}\theta_{2,t}y_{2,t}/k_{2,t}\\
w_t &= p_{1,t}(1-\theta_{m1,t}-\theta_{u1,t})y_{1,t}/n_{1,t}\\
l_t &= 1-n_{2,t}-n_{1,t}.
\end{aligned}$$

- Step 4. With leisure l_t, the wage rate w_t, and the tax on hours $\tau_{n,t}$, we can back out consumption from

$$U_c(c_t, l_t) = U_c(c_t, l_t) w_t (1 - \tau_{n,t})/(1 + \tau_{c,t}).$$

- Step 5. With corporate investments $x_{m1,t}$, $x_{u1,t}$, consumption c_t, total output y_t, and government spending g_t, we can back out total investment and investment in noncorporate capital:

$$x_t = y_t - c_t - g_t$$

$$x_{1,t} = x_{m1,t} + x_{u1,t}$$

$$x_{2,t} = x_t - x_{1,t}.$$

- Step 6. With the capital stocks and investments, we can determine next period's capital stocks:

$$k_{i,t+1} = [(1 - \delta_i) k_{i,t} + x_{i,t}]/(1 + \eta), \qquad i = m1, u1, 2.$$

We now need to compute $x_{m1,t}$, $x_{u1,t}$, and $n_{1,t}$. First, we transform the first-order equations so that all variables are stationary. The household dynamic Euler equations look as before since $1 + \eta$ will not appear there. For this problem, we have three since there are three capital stocks. The equations for capital accumulation are now

$$\tilde{k}_{i,t+1} = \left[(1 - \delta_i)\tilde{k}_{i,t} + \tilde{x}_{i,t}\right] e^{-\eta_{t+1}}/[(1 + \gamma)(1 + \eta)], \qquad i = m1, u1, 2.$$

We turn next to solving our system of difference equations. We use the static first-order conditions to eliminate n_{2t}, $\tilde{y}_{1,t}$, $\tilde{y}_{2,t}$, $\tilde{y}_t$, $p_{1,t}$, $p_{2,t}$, $r_{1,t}$, $r_{2,t}$, $\tilde{w}_t$, l_t, $\tilde{c}_t$, $\tilde{x}_t$, and $\tilde{x}_{2,t}$ in the dynamic Euler equations and the capital accumulation equations. We then linearize these equations and get a system of equations that look like

$$E_t \left\{ \Psi_1 \begin{bmatrix} \hat{k}_{m1,t+1} \\ \hat{k}_{u1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Psi_2 \begin{bmatrix} \hat{x}_{m1,t+1} \\ \hat{x}_{u1,t+1} \\ \hat{n}_{1,t+1} \end{bmatrix} + \Psi_3 \begin{bmatrix} \hat{k}_{m1,t} \\ \hat{k}_{u1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Psi_4 \begin{bmatrix} \hat{x}_{m1,t} \\ \hat{x}_{u1,t} \\ \hat{n}_{1,t} \end{bmatrix} + \Psi_5 \hat{s}_{t+1} + \Psi_6 \hat{s}_t \right\} = 0$$

$$\Phi_1 \begin{bmatrix} \hat{k}_{m1,t+1} \\ \hat{k}_{u1,t+1} \\ \hat{k}_{2,t+1} \end{bmatrix} + \Phi_2 \begin{bmatrix} \hat{x}_{m1,t+1} \\ \hat{x}_{u1,t+1} \\ \hat{n}_{1,t+1} \end{bmatrix} + \Phi_3 \begin{bmatrix} \hat{k}_{m1,t} \\ \hat{k}_{u1,t} \\ \hat{k}_{2,t} \end{bmatrix} + \Phi_4 \begin{bmatrix} \hat{x}_{m1,t} \\ \hat{x}_{u1,t} \\ \hat{n}_{1,t} \end{bmatrix} + \Phi_5 \hat{s}_{t+1} + \Phi_6 \hat{s}_t = 0$$

We are looking for a solution of the form:

$$\begin{bmatrix} \hat{x}_{m1,t} \\ \hat{x}_{u1,t} \\ \hat{n}_{1,t} \end{bmatrix} = A \begin{bmatrix} \hat{k}_{m1,t} \\ \hat{k}_{u1,t} \\ \hat{k}_{2,t} \end{bmatrix} + B\hat{s}_t.$$

The solution procedure is exactly the same as before.

To compute the expected return on corporate equities, we need to simulate the model and compute

$$r^e_{t,t+1} = \frac{[N_{t+2}V_{t+1}(X_{t+1}) + d_{t+1}(X_{t+1})N_{t+1}]}{V_t(X_t)N_{t+1}}.$$

Therefore, we want to compute the following objects:

- Per-capita Value added of sector 1: $p_{1,t}y_{1,t}$

  $=$ NIPA Corporate VA $-$ Corporate-non-property-tax IBT $+x_{u1}$

  (1) Wages: $w_t n_{1,t}$ (=NIPA corporate wages)

  (2) Measured depreciation: $\delta_{m1}k_{m1,t}$ (=NIPA corporate depreciation)

  (3) Measured property tax $\tau_{k1,t}k_{m1,t}$ (= NIPA, part of corporate IBT):

  (4) Profits: $(r_{m1,t} - \delta_{m1} - \tau_{k1,t})k_{m1,t} + r_{u1,t}k_{u1,t}$ (=NIPA profits + $x_{u1,t}$)

  Tax: $\tau_{1,t}\left[(r_{m1,t} - \delta_{m1,t} - \tau_{k1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t}\right]$ (=NIPA taxes)

  Retained earnings: $x_{m1,t} - \delta_{m1}k_{m1,t} + x_{u1,t}$ (=NIPA r.e. + $x_{u1,t}$)

  Dividends: residual (=NIPA dividends)

- Per-capita Dividends:

$$\begin{aligned}
d_t = {} & p_{1,t}y_{1,t} - w_t n_{1,t} - \tau_{k1,t}k_{m1,t} \\
& - \tau_{1,t}\left[(r_{m1,t} - \delta_{m1,t} - \tau_{k1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t}\right] - x_{m1,t} - x_{u1,t} \\
= {} & r_{m1,t}k_{m1,t} + r_{u1,t}k_{u1,t} - \tau_{k1,t}k_{m1,t} \\
& - \tau_{1,t}\left[(r_{m1,t} - \delta_{m1,t} - \tau_{k1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t}\right] \\
& - (1+\eta)k_{m1,t+1} + k_{m1,t} - \delta_{m1}k_{m1,t} - x_{u1,t} \\
= {} & (1-\tau_{1,t})\left[(r_{m1,t} - \delta_{m1} - \tau_{k1,t})k_{m1,t} + r_{u1,t}k_{u1,t} - x_{u1,t}\right] \\
& - (1+\eta)k_{m1,t+1} + k_{m1,t}
\end{aligned}$$

- Per-capita Value of corporate equities: $V_t = k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}$
- Return on corporate equities:

$$
\begin{aligned}
r^e_{t,t+1} = \Big[&(1+\eta)k_{m1,t+2} + (1+\eta)(1-\tau_{1,t+1})k_{u1,t+2} \\
&+ (1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1})k_{m1,t+1} \\
&+ (1-\tau_{1,t+1})(r_{u1,t+1}k_{u1,t+1} - x_{u1,t+1}) - (1+\eta)k_{m1,t+2} + k_{m1,t+1}\Big] \\
&\quad /(k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}) - 1 \\
= \Big[&\left\{(1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1}) + 1\right\} k_{m1,t+1} \\
&+ (1-\tau_{1,t+1})(r_{u1,t+1}k_{u1,t+1} - x_{u1,t+1}) + (1+\eta)(1-\tau_{1,t+1})k_{u1,t+2}\Big] \\
&\quad /(k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}) - 1 \\
= \Big[&\left\{(1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1}) + 1\right\} k_{m1,t+1} \\
&+ (1-\tau_{1,t+1})(r_{u1,t+1} - \delta_{u1} + 1)k_{u1,t+1}\Big] \\
&\quad /(k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}) - 1
\end{aligned}
$$

- After-tax NIPA earnings/value:

$$
\begin{aligned}
\frac{e_{t+1}}{v_t} = &\frac{(1-\tau_{1,t+1})\left[(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1})k_{m1,t+1} + r_{u1,t+1}k_{u1,t+1} - x_{u1,t+1}\right]}{k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}} \\
= \Big[&\left\{(1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1}) + 1\right\} k_{m1,t+1} \\
&+ (1-\tau_{1,t+1})(r_{u1,t+1}k_{u1,t+1} - x_{u1,t+1}) + (1-\tau_{1,t})k_{u1,t+1}\Big] \\
&\quad /(k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}) - 1 \\
= \Big[&\left\{(1-\tau_{1,t+1})(r_{m1,t+1} - \delta_{m1} - \tau_{k1,t+1}) + 1\right\} k_{m1,t+1} \\
&+ (1-\tau_{1,t+1})(r_{u1,t+1} - \delta_{u1} + 1)k_{u1,t+1} \\
&- (1-\tau_{1,t+1})(1+\eta)k_{u1,t+2} + (1-\tau_{1,t})k_{u1,t+1}\Big] \\
&\quad /(k_{m1,t+1} + (1-\tau_{1,t})k_{u1,t+1}) - 1
\end{aligned}
$$

7.3.3. Example

NIPA Incomes	Value	Formulas for Model
Value Added, Noncorporate	.473	$r_2 k_2 + w n_2$
Wages	.2554	$w n_2$
Depreciation	.118	$\delta_2 k_2$
Property Tax	.012	$\tau_{k,2} k_2$
Profits	.0876	$(r_2 - \delta_2 - \tau_{k,2}) k_2$
Value Added, corporate	.54	$r_{m1} k_{m1} + r_{u1} k_{u1} + w n_1 - x_{u1}$
Wages	.378	$w n_1$
Depreciation	.069	$\delta_1 k_{m1}$
Property Tax	.02	$\tau_{k,1} k_{m1}$
Profits	.073	$(r_{m1} - \delta_{m1} - \tau_{k,1}) k_{m1} + r_{u1} k_{u1} - x_{u1}$
Profits tax	.026	$\tau_1 \{(r_{m1} - \delta_{m1} - \tau_{k,1}) k_{m1} + r_{u1} k_{u1} - x_{u1}\}$