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Measurement with Minimal Theory*

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ABSTRACT

A central debate in applied macroeconomics is whether statistical tools that use minimal identifying assumptions are useful for isolating promising models within a broad class. In this paper, I compare three statistical models—a vector autoregressive moving average (VARMA) model, an unrestricted state space model, and a restricted state space model—that are all consistent with the same prototype business cycle model. The business cycle model is a prototype in the sense that many models, with various frictions and shocks, are observationally equivalent to it. The statistical models I consider differ in the amount of a priori theory that is imposed, with VARMAs imposing minimal assumptions and restricted state space models imposing the maximal. The objective is to determine if it is possible to successfully uncover statistics of interest for business cycle theorists with sample sizes used in practice and only minimal identifying assumptions imposed. I find that the identifying assumptions of VARMAs and unrestricted state space models are too minimal: The range of estimates are so large as to be uninformative for most statistics that business cycle researchers need to distinguish alternative theories.

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1. Introduction

A central debate in applied macroeconomics is whether statistical tools that use minimal identifying assumptions are useful for isolating promising models within a broad class. In this paper, I compare three statistical models—a vector autoregressive moving average (VARMA) model, an unrestricted state space model, and a restricted state space model—that are all consistent with the same prototype business cycle model. The business cycle model is a prototype in the sense that many models, with various frictions and shocks, are observationally equivalent to it. The statistical models I consider differ in the amount of a priori theory that is imposed, with VARMAs imposing minimal assumptions and restricted state space models imposing the maximal. The objective is to determine if it is possible to successfully uncover statistics of interest for business cycle theorists with sample sizes used in practice and only minimal identifying assumptions imposed.

I find that the identifying assumptions of VARMAs and unrestricted state space models are too minimal for practical sample sizes: The range of estimates are so large as to be uninformative for most statistics that business cycle researchers need to distinguish alternative theories. I demonstrate this by simulating 1000 datasets and applying the method of maximum likelihood to the different statistical representations for the same data. The sample sizes are two hundred periods, which is typical for real applications. The parameter estimates are used to construct standard statistics analyzed in the business cycle literature. They include impulse responses, variance decompositions, and second moments of filtered nonstationary series. Not surprising, the largest ranges are found for conditional moments such as impulse responses and variance decompositions.

In Section 2, I lay out the prototype growth model. Section 3 summarizes the three representations I use when applying maximum likelihood. Section 4 discusses the statistics computed using the maximum likelihood estimates. Section 5 concludes.

2. The Prototype Model

I use a prototype growth model as the data generating process for this study. The model is a prototype in the sense that a large class of models, including those with various types of frictions and various sources of shocks, are equivalent to a growth model with time-varying *wedges* that distort the equilibrium decisions of agents operating in otherwise competitive markets. (See Chari et al. 2006.) These wedges look like time-varying productivity, labor income taxes, and investment taxes. Since many models map into the same configuration of wedges, identifying one particular configuration does not uniquely identify a model; rather it identifies a whole class of models. Thus, the results are not specific to any one detailed economy.

Households in the economy maximize expected utility over per capita consumption c_t and per capita labor l_t,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(c_t(1-l_t)^{\psi})^{1-\sigma} - 1}{(1-\sigma)} \right] N_t$$

subject to the budget constraint and the capital accumulation law,

$$c_t + (1+\tau_{xt})x_t = (1-\tau_{lt})w_t l_t + r_t k_t + T_t$$

$$(1+g_n)k_{t+1} = (1-\delta)k_t + x_t$$

where k_t denotes the per capita capital stock, x_t per capita investment, w_t the wage rate, r_t the rental rate on capital, β the discount factor, δ the depreciation rate of capital, N_t the population with growth rate equal to $1+g_n$, and T_t the per capita lump-sum transfers. The series τ_{lt} and τ_{xt} are stochastic and stand in for time-varying distortions that affect the households' intratemporal and intertemporal decisions. Chari et al. (2006) refer to τ_{lt} as the *labor wedge* and τ_{xt} as the *investment wedge*.

The firms' production function is $F(K_t, Z_t L_t)$ where K and L are aggregate capital and labor inputs and Z_t is a labor-augmenting technology parameter which is assumed to be stochastic. Chari et al. (2006) call Z_t the *efficiency wedge* and demonstrate an equivalence between the prototype model with time-varying efficiency wedges and several detailed economies with underlying frictions that cause factor inputs to be used inefficiently. Here, I assume that the process for $\log Z_t$ is a unit-root with innovation $\log z_t$.[1] The process for the exogenous state vector $s_t = [\log z_t, \tau_{lt}, \tau_{xt}]'$ is[2]

$$\begin{aligned} s_t &= P_0 + P s_{t-1} + Q\varepsilon_t \\ &= \begin{bmatrix} g_z \\ (1-\rho_l)\tau_l \\ (1-\rho_x)\tau_x \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho_l & 0 \\ 0 & 0 & \rho_x \end{bmatrix} s_{t-1} + \begin{bmatrix} \sigma_z & 0 & 0 \\ 0 & \sigma_l & 0 \\ 0 & 0 & \sigma_x \end{bmatrix} \varepsilon_t. \end{aligned} \tag{2.1}$$

Approximate equilibrium decision functions can be computed by log-linearizing the first-order conditions and applying standard methods. (See, for example, Uhlig 1999.) The equilibrium decision function for capital has the form

$$\begin{aligned} \log \hat{k}_{t+1} &= \gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_{lt} + \gamma_x \tau_{xt} + \gamma_0 \\ &\equiv \gamma_k \log \hat{k}_t + \gamma_s' s_t + \gamma_0 \end{aligned} \tag{2.2}$$

where $\hat{k}_t = k_t / Z_{t-1}$ is detrended capital. From the static first-order conditions, I also derive decision functions for output, investment, and labor which I use later, namely,

$$\log \hat{y}_t = \phi_{yk} \log \hat{k}_t + \phi_{ys}' s_t \tag{2.3}$$

$$\log \hat{x}_t = \phi_{xk} \log \hat{k}_t + \phi_{xs}' s_t \tag{2.4}$$

$$\log l_t = \phi_{lk} \log \hat{k}_t + \phi_{ls}' s_t \tag{2.5}$$

where $\hat{y}_t = y_t / Z_t$ and $\hat{x}_t = x_t / Z_t$.

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[1] In a separate appendix, I provide a summary of how all results change when I assume technology is $Z_t = z_t (1 + g_z)^t$ with $\log z_t$ equal to an AR(1) process.

[2] The assumption that the shocks are orthogonal is unrealistic for many actual economies. Adding correlations make it more difficult for atheoretical approaches.

2.1. Observables

In all representations later, I assume that the economic modeler has data on per capita output, labor, and investment. Because output and investment grow over time, the vector of observables is taken to be

$$Y_t = [\,\Delta \log y_t/l_t \quad \log l_t \quad \log x_t/y_t\,]'\,.$$

The elements of Y are: the growth rate of log labor productivity, the log of the labor input, and the log of the investment share. All elements of Y are stationary.

For the prototype model, these observables can be written as functions of $S_t = [\log \hat{k}_t,\ s_t,\ s_{t-1}, 1]'$. To see this, note that the change in log productivity is a function of the state today $(\log \hat{k}_t, s_t, 1)$ and the state yesterday $(\log \hat{k}_{t-1}, s_{t-1}, 1)$. The capital stock at the beginning of the last period $\log \hat{k}_{t-1}$ can be written in terms of $\log \hat{k}_t$ and s_{t-1} by (2.2). The other observables depend only on today's state $(\log \hat{k}_t, s_t, 1)$. Thus, all of the observables can be written as a function of S_t.

3. Three Statistical Representations

I use the form of decision functions for the prototype model to motivate three different but related statistical representations of the economic time series.

3.1. A Restricted State Space Model

The state space model for the prototype model has the form

$$\begin{aligned} S_{t+1} &= A(\Theta)S_t + B(\Theta)\varepsilon_{t+1}, \quad E\varepsilon_t\varepsilon_t' = I \\ Y_t &= C(\Theta)S_t \end{aligned} \tag{3.1}$$

where the parameter vector is

$$\Theta = [i, g_n, g_z, \delta, \theta, \psi, \sigma, \tau_l, \tau_x, \rho_l, \rho_x, \sigma_l, \sigma_x]'.$$

Here, i is the interest rate and is used to set the discount factor $\beta = \exp(g_z)^\sigma/(1+i)$. I use Θ to compute an equilibrium and then construct

$$A(\Theta) = \begin{bmatrix} \gamma_k & \gamma_s' & 0 & \gamma_0 \\ 0 & P & 0 & P_0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B(\Theta) = \begin{bmatrix} 0 \\ Q \\ 0 \\ 0 \end{bmatrix}$$

$$C(\Theta) = \begin{bmatrix} (\phi_{yk} - \phi_{lk})(1 - 1/\gamma_k) & \phi_{lk} & \phi_{xk} - \phi_{yk} \\ \phi_{ys}' - \phi_{ls}' + \mathbf{1}' & \phi_{ls}' & \phi_{xs}' - \phi_{ys}' \\ -\phi_{ys}' + \phi_{ls}' + (\phi_{yk} - \phi_{lk})\gamma_s'/\gamma_k & 0 & 0 \\ (\phi_{yk} - \phi_{lk})\gamma_0/\gamma_k & \phi_{l0} & \phi_{x0} - \phi_{y0} \end{bmatrix}'$$

where $\mathbf{1}$ is a vector with 1 in the first element and zeros otherwise.

Estimates $\hat{\Theta}$ are found by applying the method of maximum likelihood. The exact likelihood function is computed using a Kalman filter algorithm. (See, for example, Hamilton 1994.)

For the restricted state space model, I consider three sets of restrictions on the parameter space. In what I refer to as the “loose constraints” case, I assume that the parameters in Θ can take on any value as long as an equilibrium can be computed. In what I refer to as the “modest constraints” case, I assume that the parameters in Θ are constrained to be economically plausible. Finally, I consider a “tight constraints” case with some parameters fixed during estimation. The parameters that are fixed are those that are least controversial for business cycle theorists. They are the interest rate i, the growth rates g_n and g_z, the depreciation rate δ, the capital share θ, and the mean tax rates τ_l and τ_x. In the tight-constraints case, I only estimate the parameters affecting key elasticities, namely, ψ and σ, and parameters affecting the stochastic processes for the shocks. There is no consensus on the values for these parameters.

3.2. An Unrestricted State Space Model

In the restricted state space model, all cross-equations restrictions are imposed on the state space model. This necessitates making many assumptions about the economic environment. Suppose instead that I assume only that the state of the economy evolves according to (2.1) and (2.2), and that decisions take the form of (2.3)-(2.5).

In this case, I need not provide specific details of preferences and technologies. I do, however, need to impose some minimal restrictions that imply the state space is identified. Let $\bar{S}_t = [\log \bar{k}_t, \bar{s}_t, \bar{s}_{t-1}]'$ where

$$\begin{aligned}
\log \bar{k}_t &= (\log \hat{k}_t - \log \hat{k})/(\gamma_z \sigma_z) \\
\log \bar{z}_t &= (\log z_t - \log z)/\sigma_z \\
\bar{\tau}_{lt} &= (\tau_{lt} - \tau_l)/\sigma_l \\
\bar{\tau}_{xt} &= (\tau_{xt} - \tau_x)/\sigma_x
\end{aligned}$$

and $\bar{s}_t = [\log \bar{z}_t, \bar{\tau}_{lt}, \bar{\tau}_{xt}]$. Then the unrestricted state space model can be written

$$\begin{aligned}
\bar{S}_{t+1} &= A_u(\Gamma)\bar{S}_t + B_u \varepsilon_{t+1}, \quad E\varepsilon_t \varepsilon_t' = I \\
Y_t &= C_u(\Gamma)\bar{S}_t
\end{aligned} \tag{3.2}$$

with

$$A_u(\Gamma) = \begin{bmatrix} \gamma_k & 1 & \tilde{\gamma}_l & \tilde{\gamma}_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 00 & 0 & 0 & \\ 0 & 0 & \rho_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $C_u(\Gamma)$ unrestricted (except for zero coefficients on $\bar{s}_{t-1}$ in the second and third rows). The (1,3) element of $A_u(\Gamma)$ is $\tilde{\gamma}_l = \gamma_l \sigma_l/(\gamma_z \sigma_z)$. The (1,4) element is $\tilde{\gamma}_x = \gamma_x \sigma_x/(\gamma_z \sigma_z)$.

The vector to be estimated, Γ, is therefore given by

$$\Gamma = [\gamma_k, \tilde{\gamma}_l, \tilde{\gamma}_x, \rho_l, \rho_x, \mathrm{vec}(C_u)']$$

where the $\mathrm{vec}(C_u)'$ includes only the elements that are not a priori set to 0. As in the case of the restricted state space model, estimates are found by applying the method of maximum likelihood. From this, I get $\hat{\Gamma}$.

Proposition 1. The state space model (3.2) is identified.

Proof. Applying the results of Wall (1984),[3] if (A_u^1, B_u^1, C_u^1) and (A_u^2, B_u^2, C_u^2) are observationally equivalent state space representations, then they are related by $A_u^2 = T^{-1}A_u^1 T$, $B_u^2 = T^{-1}B_u^1$, and $C_u^2 = C_u^1 T$. Identification obtains if the only matrix T satisfying these equations is $T = I$. It is simple algebra to show that this is the case for the unrestricted state space model (3.2).

3.3. A Vector Autoregression Moving Average Model

Starting from the state space representation (3.1), the moving average for the prototype model with observations Y is easily derived by recursive substitution. In particular, it is given by

$$Y_t = CB\varepsilon_t + CAB\varepsilon_{t-1} + CA^2B\varepsilon_{t-2} + \dots. \tag{3.3}$$

Assume that CB is invertible and let $e_t = CB\varepsilon_t$. Then I can rewrite (3.3) as

$$\begin{aligned} Y_t &= e_t + CAB(CB)^{-1}e_{t-1} + CA^2B(CB)^{-1}e_{t-2} + \dots \\ &\equiv e_t + C_1 e_t + C_2 e_{t-2} + \dots. \end{aligned}$$

Assuming the moving average is invertible, Y can also be represented as an infinite-order VAR,

$$Y_t = B_1 Y_{t-1} + B_2 Y_{t-2} + \dots + e_t \tag{3.4}$$

[3] See Burmeister, Wall, and Hamilton (1986), Proposition 2.

where $B_j = C_j - B_1 C_{j-1} - \ldots B_{j-1} C_1$.

Proposition 2. For the prototype economy, the implied VAR in (3.4) has the property that $M = B_j B_{j-1}^{-1}$ and therefore can be represented as a vector autoregressive moving average model of order (1,1), namely,

$$Y_t = (B_1 + M) Y_{t-1} + e_t - M e_{t-1}, \quad E e_t e_t' = \Sigma \tag{3.5}$$

with $\Sigma = CBB'C'$.

Proof. See Chari et al. (2005).

The elements of matrices B_1, M, and Σ can be estimated via maximum likelihood. To ensure stationarity and invertibility, I reparameterize the VARMA as described in Ansley and Kohn (1986). To ensure that the matrices are statistically identifiable, I also need to check that B_1 has nonzero elements and that $[B_1 + M, M]$ has full rank. (See Hannan 1976.)

I now have three statistical representations that are consistent with the prototype model. The VARMA(1,1) which imposes very minimal theory, the unrestricted state space model which imposes a little more structure, and the restricted state space model that makes explicit use of the details of the underlying model and imposes these in cross-equation restrictions. In the next section I estimate the parameters of these models and use the results to construct statistics of interest for business cycle theorists. I compare the sampling properties of the three statistical representations to see how much can be learned from each.

4. Results

Business cycle theorists use impulse response functions, variance decompositions, autocorrelations, and cross-correlations to determine which classes of economic models are most promising. In this section, I consider how much can be learned about these statistics from the three statistical representations that are consistent with my prototype model. If the sample size is infinite, all statistical procedures will uncover the true statistics because none is misspecified. But, in practice, sample sizes are no greater than two hundred periods. Thus, I draw simulations of length 200, the length typically used in practice.

Specifically, I draw 1000 simulated datasets for the prototype economy and, for each draw, estimate parameters for the three statistical representations. In all cases, the parameters of the underlying economy, Θ, are fixed. They are set at

$$\Theta = [.01, .0025, .005, .015, .33, 1.8, 1.0, .25, .0, .95, .95, 1, 1, 1]'$$

and correspond to quarterly frequencies. I assume that the parameter constraints used in the "modest constraints" case of the restricted state space model are

$$\begin{aligned}&[.0075, 0, .0025, 0, .25, .01, .01, .15, -.1, -1, -1, 0, 0, 0]\\ &\qquad < \hat{\Theta} < [.0125, .0075, .0075, .025, .45, 10, 10, .35, .1, 1, 1, 10, 10, 10].\end{aligned}$$

This implies an annual rate of interest between 3 and 5 percent; an annual growth rate of population between 0 and 3 percent; an annual growth rate of technology between 1 and 3 percent; an annual depreciation rate between 0 and 10 percent; a capital share between 25 and 45 percent; ψ and σ between 0.01 and 10; the mean labor wedge between 0.25 and 0.35; the mean investment wedge between -0.1 and 0.1; serial correlation coefficients between -1 and 1; and standard deviations of the shocks between 0 and 10 percent.

In the case of the restricted state space model, the estimation yields $\hat{\Theta}$ which can be used to construct $(\hat{A}, \hat{B}, \hat{C})$ for (3.1). In the case of the unrestricted state space, the estimation yields $\hat{\Gamma}$ which can be used to construct $(\hat{A}_u, \hat{C}_u)$ in (3.2). In the case of the VARMA model, the estimation yields $(\hat{B}_1, \hat{M}, \hat{\Sigma})$ in (3.5). Given these parameter estimates, I then construct the statistics of interest.

The first set of statistics are impulse responses of the three observables—growth in labor productivity, the log of labor, and the log of the investment share—to 1 percent shocks in each of the three shocks in ε_t. In the restricted state space model, the impact of the shock is summarized by the elements of CB. Similarly, the impact responses are summarized by $C_u B_u$ for the unrestricted state space model. For the VARMA, one needs additional information to identify CB from the variance-covariance $\Sigma = (CB)(CB)'$. A typical assumption made in the literature to identify the first column of CB is to assume that demand shocks have no long run effect on labor productivity. This assumption allows me to infer the first column of CB. (See Chari et al. 2005.) However, it does not imply anything for the relative impacts of ε_{lt} and ε_{xt}.

In Table 1, I report the impact coefficients of the impulse responses. The first row is the true value. For example, in the model, labor rises by 0.27 percent in response to a shock to the efficiency wedge and falls by 1.52 percent in response to a shock to the labor wedge. In the next three rows, I report statistics based on the restricted state space model with varying constraints. The last two rows are the results for the unrestricted state space model and the VARMA(1,1).

The results show a huge disparity between the models with minimal identifying assumptions—represented by the VARMA model and the unrestricted state space model—and maximal identifying assumptions—represented by the restricted state space model with tight constraints. For example, based on estimates of the VARMA model, 95 percent of the responses of productivity growth to a technology shock are in the range of -0.7 percent to .85 percent. Ninety-five percent of the responses of labor to a technology shock are in the range of -1.47 percent to 1.74 percent. Ninety-five percent of the responses of the investment share to a technology shock are in the range of -2.59 percent to 3.75 percent. The range of these estimates is too large to be informative for business cycle theorists.

Comparing the modest and tight constraints case for the restrictive state space model, I find that these specifications yield very similar results. The difference in estimation was the treatment of many parameters for which there is a lot of consensus, such as the capital share. In the restricted case they were fixed and in the modest constraints case they were estimated, but had economically plausible constraints. When I allow all of the parameters to be completely free, I find that for some statistics the ranges do get significantly larger. For example, one can see a significant difference in the responses of labor and the investment share.

What Table 1 also shows is that even when there is a lot of theory imposed, there can be a wide range of estimates for some statistics. For example, the impact coefficient for the response of the investment share to the labor wedge shock shows that 95 percent of the responses are between -0.95 percent and -2.77 percent, which is a wide range.

Table 2 shows results for the variance decompositions. The ordering of results is the same as in Table 1, with the most restrictive appearing first and the least appearing last. Again the striking aspect of the results is how uninformative the unrestricted state space model and VARMA model are. The means of the VARMA results for the technology shock are very close to the truth but the range is close to [0,100], which is completely uninformative.

The third set of statistics are very common in the real business cycle literature that typically reports statistics for HP-filtered time series. Specifically, for each statistical representation and each set of parameter estimates, I simulate 500 time series for output, labor, and investment of length 200. In each case, the output and investment data are filtered because they are nonstationary. I then take averages of standard deviations, autocorrelations, and cross-correlations over the 500 simulations. This is done for each model and for each of the 1000 MLE parameter vectors. These are the statistics reported in Table 3.

Notice that range of estimates is small for all models in this case. For example, in all cases, the distribution of cross-correlations of output and labor has a mean of 0.89 and the largest range of estimates is [.85,.92]. Perhaps this is not too surprising given that we do not need all of the details of a model to get an accurate prediction for unconditional moments.

The final set of statistics is related to those reported in Table 2. In Table 4, I report the variance decompositions for the HP-filtered data.[4] As before, the range of estimates for the unrestricted state space model and the VARMA model are so large that they are uninformative. In the restricted state space model, the estimates for the technology shock are very informative. This is true even for labor and investment, whose variation depends little on technology shocks. The restricted state space estimates for the labor shock imply that it contributes significantly to all three variables. The restricted state space estimates for the investment shock are least informative, but still imply that ε_x has a big effect on investment.

5. Conclusion

In this paper, I conduct a simple small-sample study. I ask how much can business cycle

[4] This is a similar exercise to that done in Table 2 but is included for easy comparison to estimates in the business cycle literature.

theorists learn from actual time series if they impose very little theory when applying their statistical methods. The answer is very little.

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