NOTES: Measurement with Minimal Theory

by Ellen McGrattan

1. RBC Model with Unit-Root Technology

Consider an economy with households, firms, and the government. The representative household chooses consumption, investment, and labor to solve the following maximization problem:

$$\begin{aligned}
&\max_{\{c_t, x_t, l_t\}} E \sum_{t=0}^{\infty} \beta^t U(c_t, 1-l_t) N_t \\
\text{subject to} \quad & c_t + (1+\tau_{xt}) x_t = r_t k_t + (1-\tau_{lt}) w_t l_t + tr_t \\
& N_{t+1} k_{t+1} = [(1-\delta) k_t + x_t] N_t \\
& c_t, x_t \geq 0 \quad \text{in all states}
\end{aligned}$$

taking processes for the rental rate, wage rate, the tax rates, and transfers as given. The representative firm solves a simple static problem at t:

$$\max_{\{K_t, L_t\}} \; F(K_t, Z_t L_t) - r_t K_t - w_t L_t.$$

The government sets rates of taxes and transfers in such a way that their budget constraint at t, namely,

$$G_t + N_t tr_t = \tau_{lt} w_t l_t N_t + \tau_{xt} N_t x_t$$

is satisfied. In equilibrium, the following conditions must hold:

$$\begin{aligned}
N_t(c_t + x_t) + G_t &= F(K_t, Z_t L_t) \qquad (1.1) \\
N_t k_t &= K_t \\
N_t l_t &= L_t.
\end{aligned}$$

1.1. First-order conditions

I now derive first-order conditions in this economy. The Lagrangian for the household optimization problem is given by

$$\begin{aligned}\mathcal{L} = E \sum_t \beta^t N_t \Big\{ & U(c_t, 1-l_t) \\ & + \mu_t \Big\{ r_t k_t + (1-\tau_{lt}) w_t l_t + tr_t - c_t - (1+\tau_{xt}) x_t \Big\} \\ & + \lambda_t \Big\{ (1-\delta) k_t + x_t - (1+g_n) k_{t+1} \Big\} \Big\}\end{aligned}$$

Here, I am assuming that the investment decision will be interior.

The relevant first-order conditions are found by taking derivatives of $\mathcal{L}$ with respect to c_t, l_t, x_t, and k_{t+1}:

$$\begin{aligned}0 &= U_1(c_t, 1-l_t) - \mu_t \\ 0 &= -U_2(c_t, 1-l_t) + \mu_t (1-\tau_{lt}) w_t \\ 0 &= \mu_t (1+\tau_{xt}) + \lambda_t = 0 \\ 0 &= -(1+g_n)\lambda_t + E_t\{\mu_{t+1} r_{t+1} + \lambda_{t+1}(1-\delta)\}\end{aligned}$$

Eliminating multipliers yields:

$$\frac{U_2(c_t, 1-l_t)}{U_1(c_t, 1-l_t)} = (1-\tau_{lt}) w_t \tag{1.2}$$

$$(1+\tau_{xt}) U_1(c_t, 1-l_t) = \beta E_t \Big[U_1(c_{t+1}, 1-l_{t+1}) \big\{ r_{t+1} + (1-\delta)(1+\tau_{xt+1}) \big\} \Big]. \tag{1.3}$$

In addition, there are first-order conditions for the firm's static problem. These are

$$r_t = F_1(K_t, Z_t L_t) \tag{1.4}$$

$$w_t = F_2(K_t, Z_t L_t) Z_t. \tag{1.5}$$

Finally, I have a resource constraint given by (1.1).

From here on, I make the following functional form assumptions and auxiliary choices:

$$F(k, l) = k^\theta l^{1-\theta} \tag{1.6}$$

$$U(c, 1-l) = (c(1-l)^\psi)^{1-\sigma}/(1-\sigma) \tag{1.7}$$

$$\tau_{kt} = \tau_{ct} = 0$$

$$s_t = [\log z_t, \tau_{lt}, \tau_{xt}, \log \hat{g}_t]'$$

$$s_{t+1} = P_0 + P s_t + Q \eta_{s,t+1}, \quad \eta_s \sim N(0_{4\times 1}, I_{4\times 4}). \tag{1.8}$$

If I substitute the choices (1.6)-(1.7) into (1.1) and (1.2)-(1.5), then substitute the equilibrium rates r_t and w_t into (1.2) and (1.3), I have:

$$N_t(c_t + g_t) + N_{t+1}k_{t+1} - (1-\delta)N_t k_t = (N_t k_t)^{\theta}(Z_t N_t l_t)^{1-\theta} \tag{1.9}$$

$$\frac{\psi c_t}{1-l_t} = (1-\tau_{lt})(1-\theta)(N_t k_t)^{\theta} Z_t^{1-\theta}(N_t l_t)^{-\theta} \tag{1.10}$$

$$\begin{aligned}(1+\tau_{xt})c_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ = \beta E_t\big[c_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}\big\{\theta(N_{t+1}k_{t+1})^{\theta-1}(Z_{t+1}N_{t+1}l_{t+1})^{1-\theta} \\ + (1-\delta)(1+\tau_{xt+1})\big\}\big].\end{aligned} \tag{1.11}$$

1.2. Log-linear computation

I first normalize the variables as follows:

$$\hat{c}_t = c_t/Z_t, \hat{x}_t = x_t/Z_t, \hat{g}_t = g_t/Z_t, \hat{y}_t = y_t/Z_t, \hat{k}_t = k_t/Z_{t-1}.$$

Using the functional forms for F and U in (1.6) and (1.7), respectively, the equilibrium rental and wage rates are:

$$\begin{aligned} r_t &= \theta K_t^{\theta-1}(Z_t L_t)^{1-\theta} = \theta \hat{k}_t^{\theta-1}(z_t l_t)^{1-\theta} \\ w_t &= (1-\theta)K_t^{\theta}(Z_t L_t)^{-\theta} Z_t = (1-\theta)\hat{k}_t^{\theta}(z_t l_t)^{-\theta} Z_t. \end{aligned}$$

This implies the following first-order conditions

$$\hat{c}_t + \hat{g}_t + (1+g_n)\hat{k}_{t+1} - (1-\delta)z_t^{-1}\hat{k}_t = \hat{y}_t = \hat{k}_t^{\theta} l_t^{1-\theta} z_t^{-\theta} \tag{1.12}$$

$$\frac{\psi \hat{c}_t}{1-l_t} = (1-\tau_{lt})(1-\theta)\hat{k}_t^{\theta}(z_t l_t)^{-\theta} \tag{1.13}$$

$$\begin{aligned}(1+\tau_{xt})\hat{c}_t^{-\sigma}(1-l_t)^{\psi(1-\sigma)} \\ = \beta z_{t+1}^{-\sigma} E_t \hat{c}_{t+1}^{-\sigma}(1-l_{t+1})^{\psi(1-\sigma)}[\theta \hat{k}_{t+1}^{\theta-1}(z_{t+1}l_{t+1})^{1-\theta} + (1-\delta)(1+\tau_{xt+1})].\end{aligned} \tag{1.14}$$

Next, I compute the steady state of the system for constant values for z, the taxes, and government spending:

$$\begin{aligned} \hat{k}/l &= \left(\frac{(1+\tau_x)(1-\beta z^{-\sigma}(1-\delta))}{\beta z^{-\sigma}\theta z^{1-\theta}}\right)^{1/(\theta-1)} \\ \hat{c} &= \left[(\hat{k}/l)^{\theta-1} z^{-\theta} - (1+g_n) + (1-\delta)z^{-1}\right]\hat{k} - \hat{g} = \xi_1 \hat{k} - \hat{g} \\ \hat{c} &= \left[(1-\tau_l)(1-\theta)(\hat{k}/l)^{\theta} z^{-\theta}/\psi\right](1 - 1/(\hat{k}/l)\ \hat{k}) = \xi_2 - \xi_3 \hat{k} \end{aligned}$$

where the last 2 equations imply $\hat{k} = (\xi_2 + \hat{g})/(\xi_1 + \xi_3)$, $\hat{c} = \xi_1 \hat{k} - \hat{g}$, $l = (1/(\hat{k}/l))\hat{k}$.

Assume that the solution for the capital decision takes the form:

$$\log \hat{k}_{t+1} = \gamma_k \log \hat{k}_t + \gamma \, [\, \log z_t \quad \tau_{lt} \quad \tau_{xt} \quad \log \hat{g}_t \,]' + \text{constant}, \tag{1.15}$$

where γ_k is a scalar and γ is 1×4 and equal to $[\gamma_z, \gamma_l, \gamma_x, \gamma_g]$. Assume the residual from the dynamic first-order condition (1.14) can be written (after substitutions from (1.12) and (1.13)):

$$\begin{aligned} &f(E_t \log \hat{k}_{t+2}, \log \hat{k}_{t+1}, \log \hat{k}_t, \log z_{t+1}, \log z_t, \tau_{lt+1}, \tau_{lt}, \tau_{xt+1}, \tau_{xt}, \log \hat{g}_{t+1}, \log \hat{g}_t) \\ &\qquad \approx a_0 E_t \log \hat{k}_{t+2} + a_1 \log \hat{k}_{t+1} + a_2 \log \hat{k}_t + b_0 E_t s_{t+1} + b_1 s_t. \end{aligned}$$

Then the general solution algorithm is to find γ_k that solves the quadratic equation

$$a_0 \gamma_k^2 + a_1 \gamma_k + a_2 = 0,$$

and γ that solves the linear equations:

$$a_0 \gamma_k \gamma + a_0 \gamma P + a_1 \gamma + b_0 P + b_1 = 0_{1 \times 4}.$$

Note that this implies:

$$\gamma = -[(a_0 a + a_1) I_{4 \times 4} + a_0 P']^{-1} (b_0 P + b_1 I_{4 \times 4})'. \tag{1.16}$$

Once I have values for the the coefficients γ_k and γ, I can use (1.12) and (1.13) to back out c_t and l_t (either nonlinearly or by way of a log-linear approximation).

One property of the solution that I use later is the fact that $\gamma_k = -\gamma_z$. This is true because $\hat{k}_t$ is everywhere divided by z_t in the first-order conditions (1.12)-(1.13). Thus, when the first-order conditions are log-linearized, the same coefficients hit $\log(\hat{k}_t)$ and $-\log(z_t)$.

Given values for the coefficients in (1.15), I can derive expressions for labor, consumption, and investment using the static first-order conditions. In particular, I log-linearize (1.13) after substituting in for consumption from (1.12):

$$\begin{aligned} 0 \approx \psi \big\{ &\hat{k}^\theta l^{1-\theta} z^{-\theta} [\theta(\log \hat{k}_t - \log z_t) + (1-\theta) \log l_t] \\ &- (1+g_n)\hat{k} \log \hat{k}_{t+1} + (1-\delta) z^{-1} \hat{k} (\log \hat{k}_t - \log z_t) - \hat{g} \log \hat{g}_t \big\} \\ &+ (1-\theta)(1-\tau_l) \hat{k}^\theta (zl)^{-\theta} (1-l) \big\{ 1/(1-\tau_l) \, \tau_{lt} \\ &\qquad - \theta \log \hat{k}_t + \theta(\log l_t + \log z_t) + l/(1-l) \log l_t \big\}. \end{aligned}$$

which I write succinctly as

$$\log l_t = \phi_{lk} \log \hat{k}_t + \phi_{lz} \log z_t + \phi_{ll} \tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}.$$

With this equation for $\log l$, I use the production relation and the capital accumulation equation to write $\log \hat{y}$ and $\log \hat{x}$ as follows:

$$\begin{aligned}
\log \hat{y}_t &= (\theta + (1-\theta)\phi_{lk}) \log \hat{k}_t + ((1-\theta)\phi_{lz} - \theta) \log z_t \\
&\qquad + (1-\theta)[\phi_{ll}\tau_{lt} + \phi_{lg} \log \hat{g}_t + \phi_{lk'} \log \hat{k}_{t+1}] \\
&\equiv \phi_{yk} \log \hat{k}_t + \phi_{yz} \log z_t + \phi_{yl}\tau_{lt} + \phi_{yg} \log \hat{g}_t + \phi_{yk'} \log \hat{k}_{t+1} && (1.17) \\
\log \hat{x}_t &\approx (1+g_n)\hat{k}/\hat{x} \log \hat{k}_{t+1} - (1-\delta)z^{-1}\hat{k}/\hat{x}(\log \hat{k}_t - \log z_t) \\
&\equiv \phi_{xk} \log \hat{k}_t + \phi_{xz} \log z_t + \phi_{xk'} \log \hat{k}_{t+1}. && (1.18)
\end{aligned}$$

Finally, I can log-linearize (1.12) to get

$$\begin{aligned}
\log \hat{c}_t &\approx \{\hat{y}[\theta(\log \hat{k}_t - \log z_t) + (1-\theta) \log l_t] - \hat{g} \log \hat{g}_t \\
&\qquad - (1+g_n)\hat{k} \log \hat{k}_{t+1} + (1-\delta)z^{-1}\hat{k}[\log \hat{k}_t - \log z_t]\}/\hat{c} \\
&= [\theta\hat{y}/\hat{c} + (1-\theta)\phi_{lk}\hat{y}/\hat{c} + (1-\delta)\hat{k}/(\hat{c}z)] \log \hat{k}_t \\
&\qquad - [\theta\hat{y}/\hat{c} - (1-\theta)\phi_{lz}\hat{y}/\hat{c} + (1-\delta)\hat{k}/(\hat{c}z)] \log z_t \\
&\qquad + [(1-\theta)\phi_{ll}\hat{y}/\hat{c}]\tau_{lt} \\
&\qquad + [(1-\theta)\phi_{lg}\hat{y}/\hat{c} - \hat{g}/\hat{c}] \log \hat{g}_t \\
&\qquad + [(1-\theta)\phi_{lk'}\hat{y}/\hat{c} - (1+g_n)\hat{k}/\hat{c}] \log \hat{k}_{t+1} \\
&\equiv \phi_{ck} \log \hat{k}_t + \phi_{cz} \log z_t + \phi_{cl}\tau_{lt} + \phi_{cg} \log \hat{g}_t + \phi_{ck'} \log \hat{k}_{t+1}. && (1.19)
\end{aligned}$$

1.3. The Model's Moving Average

Y_t are observables. η_t are primitive shocks. Need D0, D1, etc for all possiblilities

Possible observables: $\Delta log(y/l)$, $(1-\alpha L)logl$, $log(c/y)$, $log(x/y)$. $\Delta log(y)$ $\Delta log(c)$ $\Delta log(x)$ $\Delta log(k)$

FIX OR BORROW BELOW:

2. VARs and the 2-Shock Version of the Model

2.1. The Decision Functions

Assume the economy has only two shocks and they are orthogonal: a unit root in technology $\log z$ and an AR(1) in the tax rate on labor τ. (For convenience I drop l on τ_{lt} throughout this section.) The capital decision function has the form:

$$\log \hat{k}_{t+1} = \gamma_0 + \gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_t$$

and the labor decision function can be written:

$$\begin{aligned}
\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk} \log \hat{k}_t + \phi_{lk'} \log \hat{k}_{t+1} \\
&= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk} \log \hat{k}_t + \phi_{lk'}[\gamma_0\gamma_k \log \hat{k}_t + \gamma_z \log z_t + \gamma_l \tau_t] \\
&= (\phi_{lk} + \phi_{lk'}\gamma_k) \log \hat{k}_t + (\phi_{lz} + \phi_{lk'}\gamma_z) \log z_t + (\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t.
\end{aligned}$$

These imply that output from a Cobb-Douglas production technology with capital share θ is:

$$\begin{aligned}
\log \hat{y}_t &= \theta(\log \hat{k}_t - \log z_t) + (1-\theta)\log l_t \\
&= (\theta + (1-\theta)\phi_{lk}) \log \hat{k}_t - (\theta - (1-\theta)\phi_{lz}) \log z_t + (1-\theta)\phi_{ll}\tau_t \\
&\qquad + (1-\theta)\phi_{lk'} \log \hat{k}_{t+1} \\
&= (\theta + (1-\theta)(\phi_{lk} + \phi_{lk'}\gamma_k)) \log \hat{k}_t - (\theta - (1-\theta)(\phi_{lz} + \phi_{lk'}\gamma_z)) \log z_t \\
&\qquad + (1-\theta)(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t
\end{aligned}$$

I can write the capital stock in terms of all lagged shocks as follows:

$$\begin{aligned}
\log \hat{k}_t &= \gamma_0 + \gamma_k(\gamma_0 + \gamma_k \log \hat{k}_{t-2} + \gamma_z \log z_{t-2} + \gamma_l \tau_{t-2}) + \gamma_z \log z_{t-1} + \gamma_l \tau_{t-1} \\
&= \gamma_0[1 + \gamma_k + \gamma_k^2 + \ldots] \\
&\qquad + \gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\
&\qquad + \gamma_l[\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots]
\end{aligned}$$

or in differences as follows:

$$\begin{aligned}\log \hat{k}_t - \log \hat{k}_{t-1} = \gamma_z[&\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\ &+ \gamma_l[\tau_{t-1} + (\gamma_k - 1)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}]\end{aligned}$$

or in quasi-differences as follows:

$$\begin{aligned}\log \hat{k}_t - \alpha \log \hat{k}_{t-1} = \gamma_z[&\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\ &+ \gamma_l[\tau_{t-1} + (\gamma_k - \alpha)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}]\end{aligned}$$

I can also write hours in terms of past shocks as follows:

$$\begin{aligned}\log l_t &= \phi_{lz} \log z_t + \phi_{ll}\tau_t + \phi_{lk} \log \hat{k}_t + \phi_{lk'} \log \hat{k}_{t+1} \\ &= \phi_{lz} \log z_t + \phi_{ll}\tau_t \\ &\quad + \phi_{lk}\gamma_z[\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots] \\ &\quad\quad + \phi_{lk}\gamma_l[\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots] \\ &\quad + \phi_{lk'}\gamma_z[\log z_t + \gamma_k \log z_{t-1} + \gamma_k^2 \log z_{t-2} + \ldots] \\ &\quad\quad + \phi_{lk'}\gamma_l[\tau_t + \gamma_k \tau_{t-1} + \gamma_k^2 \tau_{t-2} + \ldots] \\ &= [(\phi_{lz} + \phi_{lk'}\gamma_z) \log z_t + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_z \log z_{t-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_z \log z_{t-2} + \ldots] \\ &\quad + [(\phi_{ll} + \phi_{lk'}\gamma_l)\tau_t + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_l\tau_{t-1} + (\phi_{lk} + \phi_{lk'}\gamma_k)\gamma_k\gamma_l\tau_{t-2} + \ldots]\end{aligned}$$

where I have ignored constant terms.

I can write logged hours in differences as follows:

$$\begin{aligned}\log l_t - \log l_{t-1} &= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_t - \tau_{t-1}) \\ &\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \log \hat{k}_{t-1}) \\ &= \phi_{lz}(\log z_t - \log z_{t-1}) + \phi_{ll}(\tau_t - \tau_{t-1}) \\ &\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - 1)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\ &\quad + \phi_{lk'}\gamma_l[\tau_t + (\gamma_k - 1)\{\tau_{t-1} + \gamma_k \tau_{t-2} + \gamma_k^2 \tau_{t-3} + \ldots\}] \\ &\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - 1)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\ &\quad + \phi_{lk}\gamma_l[\tau_{t-1} + (\gamma_k - 1)\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}] \\ &= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)] \log z_{t-1} \\ &\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\ &\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\ &\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{t-2} + \gamma_k \tau_{t-3} + \gamma_k^2 \tau_{t-4} + \ldots\}\end{aligned}$$

or in quasi-difference form as follows:

$$
\begin{aligned}
\log l_t - \alpha \log l_{t-1} &= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_t - \alpha\tau_{t-1}) \\
&\quad + \phi_{lk'}(\log \hat{k}_{t+1} - \alpha \log \hat{k}_t) + \phi_{lk}(\log \hat{k}_t - \alpha \log \hat{k}_{t-1}) \\
&= \phi_{lz}(\log z_t - \alpha \log z_{t-1}) + \phi_{ll}(\tau_t - \alpha\tau_{t-1}) \\
&\quad + \phi_{lk'}\gamma_z[\log z_t + (\gamma_k - \alpha)\{\log z_{t-1} + \gamma_k \log z_{t-2} + \gamma_k^2 \log z_{t-3} + \ldots\}] \\
&\quad + \phi_{lk'}\gamma_l[\tau_t + (\gamma_k - \alpha)\{\tau_{t-1} + \gamma_k\tau_{t-2} + \gamma_k^2\tau_{t-3} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_z[\log z_{t-1} + (\gamma_k - \alpha)\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\}] \\
&\quad + \phi_{lk}\gamma_l[\tau_{t-1} + (\gamma_k - \alpha)\{\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2\tau_{t-4} + \ldots\}] \\
&= [\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t - [\alpha\phi_{lz} - \phi_{lk}\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - \alpha)] \log z_{t-1} \\
&\quad + \gamma_z(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots\} \\
&\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\alpha\phi_{ll} - \phi_{lk}\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - \alpha)]\tau_{t-1} \\
&\quad + \gamma_l(\gamma_k - \alpha)[\phi_{lk'}\gamma_k + \phi_{lk}]\{\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2\tau_{t-4} + \ldots\}
\end{aligned}
$$

I can use the expressions for output and hours to write out the change in productivity as follows:

$$
\begin{aligned}
&\log(y_t/l_t) - \log(y_{t-1}/l_{t-1}) \\
&\qquad = \log \hat{y}_t - \log \hat{y}_{t-1} + \log z_t - \log l_t - \log l_{t-1} \\
&\qquad = \log z_t + \theta(\log \hat{k}_t - \log \hat{k}_{t-1} - \log l_t + \log l_{t-1} - \log z_t + \log z_{t-1}) \\
&\qquad = (1-\theta) \log z_t + \theta \log z_{t-1} \\
&\qquad\quad - \theta(\log l_t - \log l_{t-1} - \log \hat{k}_t + \log \hat{k}_{t-1}) \\
&\qquad = (1-\theta) \log z_t + \theta \log z_{t-1} - \theta\{[\phi_{lz} + \phi_{lk'}\gamma_z] \log z_t \\
&\qquad\quad - [\phi_{lz} - (\phi_{lk} - 1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)] \log z_{t-1} \\
&\qquad\quad + \gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\qquad\quad + [\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t - [\phi_{ll} - (\phi_{lk} - 1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\
&\qquad\quad + \gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots]\} \\
&\qquad = \{1 - \theta - \theta[\phi_{lz} + \phi_{lk'}\gamma_z]\} \log z_t \\
&\qquad\quad + \theta[1 + \phi_{lz} - (\phi_{lk} - 1)\gamma_z - \phi_{lk'}\gamma_z(\gamma_k - 1)] \log z_{t-1} \\
&\qquad\quad - \theta\gamma_z(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\log z_{t-2} + \gamma_k \log z_{t-3} + \gamma_k^2 \log z_{t-4} + \ldots] \\
&\qquad\quad - \theta[\phi_{ll} + \phi_{lk'}\gamma_l]\tau_t \\
&\qquad\quad + \theta[\phi_{ll} - (\phi_{lk} - 1)\gamma_l - \phi_{lk'}\gamma_l(\gamma_k - 1)]\tau_{t-1} \\
&\qquad\quad - \theta\gamma_l(\gamma_k - 1)[\phi_{lk'}\gamma_k + \phi_{lk} - 1][\tau_{t-2} + \gamma_k\tau_{t-3} + \gamma_k^2\tau_{t-4} + \ldots]
\end{aligned}
$$

2.2. The Model's Moving Average

The moving average for the model is given by:

$$\begin{bmatrix} (1-L)\log y_t/l_t \\ (1-\alpha L)\log l_t \end{bmatrix} \equiv X_t = D_0\omega_t + D_1\omega_{t-1} + D_2\omega_{t-2} + \dots$$

where $\omega_t = [\log z_t, \tau_t]'$ and

$$D_0 = \begin{bmatrix} 1-\theta-\theta(\phi_{lz}+\phi_{lk'}\gamma_z) & -\theta(\phi_{ll}+\phi_{lk'}\gamma_l) \\ \phi_{lz}+\phi_{lk'}\gamma_z & \phi_{ll}+\phi_{lk'}\gamma_l \end{bmatrix}$$
$$D_1 = \begin{bmatrix} \theta(1+\phi_{lz}-(\phi_{lk}-1)\gamma_z-\phi_{lk'}\gamma_z(\gamma_k-1)) & \theta(\phi_{ll}-(\phi_{lk}-1)\gamma_l-\phi_{lk'}\gamma_l(\gamma_k-1)) \\ -\alpha\phi_{lz}+(\phi_{lk}+\phi_{lk'}(\gamma_k-\alpha))\gamma_z & -\alpha\phi_{ll}+(\phi_{lk}+\phi_{lk'}(\gamma_k-\alpha))\gamma_l \end{bmatrix}$$
$$D_2 = \begin{bmatrix} -\theta\gamma_z(\gamma_k-1)[\phi_{lk'}\gamma_k+\phi_{lk}-1] & -\theta\gamma_l(\gamma_k-1)[\phi_{lk'}\gamma_k+\phi_{lk}-1] \\ (\phi_{lk}+\phi_{lk'}\gamma_k)(\gamma_k-\alpha)\gamma_z & (\phi_{lk}+\phi_{lk'}\gamma_k)(\gamma_k-\alpha)\gamma_l \end{bmatrix}$$

and $D_j = \gamma_k D_{j-1}$ for $j \geq 3$.

Let $a = \phi_{lk}+\phi_{lk'}\gamma_k$ and $b = \phi_{ll}+\phi_{lk'}\gamma_l$. Also, note that $\phi_{lz} = -\phi_{lk}$ and $\gamma_z = -\gamma_k$ hold in the model economy with a unit root in technology.

$$D_0 = \begin{bmatrix} 1-\theta+\theta a & -\theta b \\ -a & b \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \theta(1-\gamma_k)(1-a) & \theta(b+(1-a)\gamma_l) \\ (\alpha-\gamma_k)a & -\alpha b+\gamma_l a \end{bmatrix}$$

$$D_2 = \begin{bmatrix} \gamma_k(1-a)\theta(1-\gamma_k) & -\gamma_l(1-a)\theta(1-\gamma_k) \\ \gamma_k a(\alpha-\gamma_k) & -\gamma_l a(\alpha-\gamma_k) \end{bmatrix}$$

and, again, $D_j = \gamma_k^{j-2}D_2$ for $j \geq 3$. Note that D_2 is singular.

If τ_t is an AR(1), it is more convenient to write the MA process in terms of $\eta_t = [\log z_t, \eta_{lt}]$ rather than in terms of ω_t. In this case,

$$X_t = D_0\eta_t+(D_0P+D_1)\eta_{t-1}+(D_0P^2+D_1P+D_2)\eta_{t-2}+(D_0P^3+D_1P^2+D_2P+D_3)\eta_{t-3}\dots.$$

I normalize the MA so it has an identity for the first coefficient. That is, I set $C_0 = I$, $C_1 = (D_0P+D_1)D_0^{-1}$, and $C_j = C_{j-1}D_0PD_0^{-1}+D_jD_0^{-1}$.

2.3. Special Property of the D's

Next, I will show that the D matrices have a special property that will be exploited when I characterize coefficients of the VAR found by regressing X_t on lags of itself. The D's for the RBC model satisfy the relation:

$$\left(\gamma_k I - (D_0 P^2 + D_1 P + D_2)(D_0 P + D_1)^{-1}\right) D_2 = 0. \tag{2.1}$$

One method of proof is to multiply all terms of the matrices in (2.1) and show that all elements are zero. I have done this but the algebra is messy.

A simpler proof is as follows. Note that

$$D_2 = \begin{bmatrix} (1-a)\theta(1-\gamma_k) \\ (\alpha-\gamma_k)a \end{bmatrix} [\,\gamma_k \quad -\gamma_l\,] \equiv gh'. \tag{2.2}$$

Thus, I can rewrite the left hand side of (2.1) as follows

$$\begin{aligned} &\left(\gamma_k I - (D_0 P^2 + D_1 P + D_2)(D_0 P + D_1)^{-1}\right) D_2 \\ &= \left[\gamma_k(gh') - (gh')(D_0 P + D_1)^{-1}(gh')\right] - \left[(D_0 P + D_1)P(D_0 P + D_1)^{-1} gh'\right]. \end{aligned} \tag{2.3}$$

I will prove that both terms in (2.3) in square brackets is equal to 2×2 zero matrices. The first step of the proof is to show that

$$(D_0 P + D_1)^{-1} g = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.4}$$

The proof of this step is trivial since the first column of D_0P+D_1 is equal to g. Substituting (2.4) into (2.3), the result (2.1) follows immediately from the fact that $h'[1,0]' = \gamma_k$ and $P[1,0]' = 0$.

2.4. VAR Coefficients

Given expressions for the D coefficients in the model MA, and thus the normalized C coefficients, I can directly write out expressions for the coefficients in the VAR of X_t regressed on lags of itself. I will denote the VAR coefficients by B_j, $j = 1, 2, \ldots$. They are related to the MA coefficients as follows:

$$B_j = C_j - B_1 C_{j-1} - B_2 C_{j-2} - \ldots B_{j-1} C_1. \tag{2.5}$$

2.5. Proposition 1: Model has infinite-order VAR

Proposition 1. The model described above has a VAR representation with coefficients B_j that satisfy

$$B_j = MB_{j-1} \tag{2.6}$$

for $j \geq 2$, with $B_1 = C_1 = (D_0 P + D_1)D_0^{-1}$. The matrix M is a 2×2 matrix with eigenvalues equal to α and

$$\frac{\gamma_k - \gamma_l a/b - \theta}{1-\theta},$$

where $a = \phi_{lk} + \phi_{lk'}\gamma_k$ and $b = \phi_{ll} + \phi_{lk'}\gamma_l$ are the coefficients on k and τ_l in the labor decision function.

Proof of Proposition 1. Choose $M = C_2 C_1^{-1} - C_1$. Using the formula (2.5) for the VAR coefficient, it is easy to show that $M = B_2 B_1^{-1}$. Therefore, $B_2 = MB_1$ holds. Consider the next coefficient. Using the formula (2.5), I have

$$\begin{aligned}
B_3 - MB_2 &= C_3 - B_1C_2 - B_2C_1 - M(C_2 - B_1C_1) \\
&= C_3 - B_1C_2 - MC_2 \\
&= C_3 - C_1C_2 - (C_2C_1^{-1} - C_1)C_2 \\
&= C_3 - C_2C_1^{-1}C_2 \\
&= C_2D_0PD_0^{-1} + \gamma_k D_2D_0^{-1} - C_2C_1^{-1}(C_1D_0PD_0^{-1} + D_2D_0^{-1}) \\
&= \gamma_k D_2D_0^{-1} - C_2C_1^{-1}D_2D_0^{-1}) \\
&= (\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= (\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1})D_2D_0^{-1} \\
&= 0
\end{aligned}$$

where the last relation follows from intermediate calculations done in Section 3. The same calculation can be done for any j^* using the fact that (2.6) holds for all $j < j^*$, namely

$$\begin{aligned}
B_j - MB_{j-1} &= C_j - B_1C_{j-1} \ldots - B_{j-1}C_1 - M(C_{j-1} - \ldots B_{j-2}C_1) \\
&= C_j - B_1C_{j-1} - MC_{j-1} \\
&= C_j - C_1C_{j-1} - (C_2C_1^{-1} - C_1)C_{j-1} \\
&= C_j - C_2C_1^{-1}C_{j-1} \\
&= C_{j-1}D_0PD_0^{-1} + D_jD_0^{-1} - C_2C_1^{-1}(C_{j-2}D_0PD_0^{-1} + D_{j-1}D_0^{-1}) \\
&= C_{j-1}D_0PD_0^{-1} + \gamma_k^{j-2}D_2D_0^{-1} - C_2C_1^{-1}(C_{j-2}D_0PD_0^{-1} + \gamma_k^{j-3}D_2D_0^{-1}) \\
&= (C_{j-1} - C_2C_1^{-1}C_{j-2})D_0PD_0^{-1} + \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= (B_{j-1} - MB_{j-2})D_0PD_0^{-1} + \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= \gamma_k^{j-3}(\gamma_k I - C_2C_1^{-1})D_2D_0^{-1} \\
&= \gamma_k^{j-3}(\gamma_k I - (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1})D_2D_0^{-1} \\
&= 0.
\end{aligned}
$$

Next, I prove that the two eigenvalues of M are $\lambda_1 = \alpha$ and $\lambda_2 = (\gamma_k - \gamma_l a/b - \theta)/(1-\theta)$. One way to do this is to write out all of the terms for matrix M and derive expressions for the trace and the determinant. The trace is equal to the sum of the eigenvalues and the determinant is equal to the product of the eigenvalues. This is 2 equations and 2 unknowns. I have done this but the algebra is messy.

A simpler proof that the eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = (\gamma_k - \gamma_l a/b - \theta)/(1-\theta)$ is as follows. Using (2.2) and the definitions of the C's in terms of the D's, I can derive the following expression for M in terms of the D's, P, and h:

$$
\begin{aligned}
M &= C_2C_1^{-1} - C_1 \\
&= (D_0P^2 + D_1P + D_2)(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1} \\
&= (D_0P + D_1)P(D_0P + D_1)^{-1} + D_2(D_0P + D_1)^{-1} \\
&\quad - (D_0P + D_1)D_0^{-1}(D_0P + D_1)(D_0P + D_1)^{-1} \\
&= D_2(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1}D_1(D_0P + D_1)^{-1} \\
&= (D_0P + D_1)[1,0]'h'(D_0P + D_1)^{-1} - (D_0P + D_1)D_0^{-1}D_1(D_0P + D_1)^{-1} \\
&= (D_0P + D_1)([1,0]'h' - D_0^{-1}D_1)(D_0P + D_1)^{-1}.
\end{aligned}
$$

Appealing to standard results in linear algebra, the eigenvalues of M are equal to the eigenvalues of the simpler matrix $[1,0]'h' - D_0^{-1}D_1$, which is equal to

$$
[1,0]'h' - D_0^{-1}D_1 = \frac{1}{1-\theta}\begin{bmatrix} \gamma_k - \theta(1 - a + a\alpha) & -\gamma_l + \theta b(1-\alpha) \\ (\gamma_k - \alpha - \theta(1-a)(1-\alpha))a/b & \alpha - \theta(a + \alpha(1-a)) - \gamma_l a/b \end{bmatrix}
$$

Taking the trace, I get

$$
\operatorname{trace}([1,0]'h' - D_0^{-1}D_1) = \alpha + \frac{\gamma_k - \gamma_l a/b - \theta}{1-\theta}. \tag{2.7}
$$

Taking the determinant, I get

$$
\det([1,0]'h' - D_0^{-1}D_1) = \alpha \times \frac{\gamma_k - \gamma_l a/b - \theta}{1-\theta}. \tag{2.8}
$$

The two equations (2.7) and (2.8) uniquely determine the two eigenvalues which are those proposed. ∎

2.6. Blanchard-Quah Identification

I now consider the procedure of Blanchard and Quah (1989) when applied to data from the 2-shock version of the model described above.

Blanchard and Quah start, as I did, with a VAR

$$X_t = B_1 X_{t-1} + B_2 X_{t-2} + \dots B_p X_{t-p} + v_t, \qquad E v_t v_t = \Omega$$

which is estimated using time series $\{X_t\}$. As described above, this implies the MA

$$X_t = v_t + C_1 v_{t-1} + C_2 v_{t-2} + \dots. \tag{2.9}$$

Some structure is needed to derive a "structural MA" with shocks that have economic interpretation. In this case, I will use the following notation for the structural MA:

$$X_t = A_0 \epsilon_t + A_1 \epsilon_{t-1} + A_2 \epsilon_{t-2} + \dots. \tag{2.10}$$

where $A_0 \epsilon_t = v_t$ and $A_j = C_j A_0$.

Because I will impose restrictions on the sums of the A's and C's, I define

$$\begin{aligned}\bar{C} &= I + C_1 + C_2 + C_3 + \dots \\ \bar{A} &= A_0 + A_1 + A_2 + A_3 + \dots \\ &= \bar{C} A_0.\end{aligned}$$

Since $A_0 \epsilon_t = v_t$, it must be the case that $A_0 E \epsilon_t \epsilon_t A_0' = \Omega$. Blanchard and Quah assume that the elements of ϵ_t are orthogonal and demand shocks do not have a long-run effect on productivity. Without loss of generality, we can normalize the magnitude of the variances of the elements of ϵ_t and assume, therefore, that

$$\begin{aligned}&A_0 A_0' = \Omega \\ &\bar{C}(1,1) A_0(1,2) + \bar{C}(1,2) A_0(2,2) = 0\end{aligned} \tag{2.11}$$

which is four equations in the four unknown elements of A_0. Condition (2.11) ensures that the demand shock does not have a long-run effect on productivity. Writing out the system of 4 equations and 4 unknowns yields:

$$\begin{aligned}\omega_{11} &= A_0(1,1)^2 + A_0(1,2)^2 \\ \omega_{12} &= A_0(1,1) A_0(2,1) + A_0(1,2) A_0(2,2) \\ \omega_{22} &= A_0(2,1)^2 + A_0(2,2)^2 \\ 0 &= \bar{C}(1,1) A_0(1,2) + \bar{C}(1,2) A_0(2,2)\end{aligned} \tag{2.12}$$

Eliminate $A_0(1,2)$ using the fact that $A_0(1,2) = -\bar{C}(1,2)A_0(2,2)/\bar{C}(1,1)$:

$$\begin{aligned}
\omega_{11} &= A_0(1,1)^2 + f^2 A_0(2,2)^2 \\
\omega_{12} &= A_0(1,1)A_0(2,1) + f A_0(2,2)^2 \\
\omega_{22} &= A_0(2,1)^2 + A_0(2,2)^2
\end{aligned}$$

where $f = -\bar{C}(1,2)/\bar{C}(1,1)$. Solve for $A_0(1,1)$ and $A_0(2,1)$:

$$A_0(1,1) = [\omega_{11} - f^2 A_0(2,2)^2]^{1/2} \tag{2.13}$$

$$A_0(2,1) = [\omega_{22} - A_0(2,2)^2]^{1/2} \tag{2.14}$$

and substitute to get:

$$\omega_{12} = f A_0(2,2)^2 + [\omega_{11} - f^2 A_0(2,2)^2]^{1/2}[\omega_{22} - A_0(2,2)^2]^{1/2}.$$

Let $\lambda = A_0(2,2)^2$ and the result is a quadratic in λ:

$$(\omega_{12} - f\lambda)^2 = (\omega_{11} - f^2\lambda)(\omega_{22} - \lambda)$$

which can be written out:

$$\omega_{12}^2 - 2f\lambda\omega_{12} + f^2\lambda^2 = \omega_{11}\omega_{22} - f^2\lambda\omega_{22} - \omega_{11}\lambda + f^2\lambda^2$$

and simplified as follows:

$$\lambda = \frac{\omega_{11}\omega_{22} - \omega_{12}^2}{\omega_{11} + f^2\omega_{22} - 2f\omega_{12}}. \tag{2.15}$$

In addition, I need to impose sign conventions since impulse responses can be either positive or negative. I will consider one sign convention for the demand shock and two different sign conventions for the technology shock.

The demand shock in our example is a shock to the tax rate on labor. For this choice of shock, I want to impose $A_0(2,2) < 0$ so that hours fall with a positive shock to the tax rate on labor. For $A_0(2,2)$, it must be the case that $A_0(2,2) = -\sqrt{\lambda}$ since λ is positive. Given $A_0(2,2)$, it immediately follows that $A_0(1,2) = f A_0(2,2)$.

2.6.1. Sign convention on $A_0(1,1)$

For the technology shock, I first consider the sign convention that productivity rises on impact in response to a positive technology shock, namely $A_0(1,1) > 0$. In this case, I need to use the positive root of $A_0(1,1)^2 = \omega_{11} - f^2\lambda$:

$$A_0(1,1) = \sqrt{\omega_{11} - f^2\lambda}.$$

Given a value for $A_0(1,1)$, I have $A_0(2,1)$ from:

$$A_0(2,1) = (\omega_{12} - f\lambda)/A_0(1,1).$$

2.6.2. Sign convention on $\bar{A}(1,1)$

The second sign convention assumes that productivity is positive in the long run so that $\bar{A}(1,1) > 0$ and therefore

$$\bar{C}(1,1)A_0(1,1) + \bar{C}(1,2)A_0(1,2) > 0.$$

This condition can also be written in terms of $A_0(1,1)$ and known parameters:

$$\bar{C}(1,1)A_0(1,1) + \bar{C}(1,2)(\omega_{1,2} - f\lambda)/A_0(1,1) > 0. \tag{2.16}$$

In this case, I choose the sign on the square root of $\omega_{11} - f^2\lambda$ so that (2.16) is satisfied.

2.6.3. Full solution

The full solution is

$$\begin{aligned}
A_0(2,2) &= -\sqrt{\lambda} \\
A_0(1,2) &= fA_0(2,2) \\
A_0(1,1) &= \text{root of } \omega_{11} - f^2\lambda \text{ satisfying sign convention} \\
A_0(2,1) &= (\omega_{12} - f\lambda)/A_0(1,1)
\end{aligned}$$

where λ is defined in (2.15) and $f = -\bar{C}(1,2)/\bar{C}(1,1)$.

2.6.4. Cholesky decomposition

In the literature, many report using the following formula for A_0:

$$A_0 = \bar{C}^{-1} L$$

where L is a lower triangular matrix such that with positive elements on the diagonal that satisfies $LL' = \bar{C}\Omega\bar{C}'$. This choice imposes the long-run restriction in (2.11) and the long-run sign convention $\bar{A}(1,1)$ automatically.

It does not impose $A_0(2,2) < 0$. However, in most cases, responses to demand shocks are not discussed.

2.7. Proposition 2: OLS Results

Proposition 1 says that the model has an infinite-lag vector autoregressive structure. The next proposition considers the outcome when OLS regressions are run with one lag. Let $V_0 = EX_tX_t'$ be the theoretical variance matrix for X_t. Let $V_1 = EX_tX_{t-1}'$ be the covariance matrix for X_t and its lag. If $ED_0\eta_t\eta_t'D_0' = \Omega$ is the theoretical variance-covariance of the model's shock vector, then

$$V_0 = \Omega + C_1\Omega C_1' + C_2\Omega C_2' + \dots \tag{2.17}$$

$$V_1 = C_1\Omega + C_2\Omega C_1' + C_3\Omega C_2' + \dots. \tag{2.18}$$

Proposition 2. Assume that a regression is run of the form

$$X_t = B_{ols}X_{t-1} + v_t, \quad Ev_tv_t' = \Omega_{ols}$$

with X_t from the RBC model. Then, the variance-covariance matrix is

$$\Omega_{ols} = V_0 - V_1V_0^{-1}V_1' \tag{2.19}$$

$$= \Omega + M\Omega M' - M\Omega V_0^{-1}\Omega M' \tag{2.20}$$

where $M = C_2C_1^{-1} - C_1$ and the inverse of the sum of MA coefficients is

$$\begin{aligned}\bar{C}_{ols}^{-1} &= I - B_{ols}\\ &= \bar{C}^{-1} + M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1}.\end{aligned}$$

In other words, the OLS matrices Ω_{ols} and $\bar{C}_{ols}$ are not equal to their theoretical counterparts, Ω and $\bar{C}$.

Proof of Proposition 2. The relation (2.19) follows from the standard projection formulas,

$$\begin{aligned} B_{ols} &= (EX_t X_{t-1})(EX_{t-1}X'_{t-1})^{-1} = V_1 V_0^{-1} \\ Ev_t v'_t &= E(X_t - B_{ols}X_{t-1})(X_t - B_{ols}X_{t-1})' = V_0 - V_1 V_0^{-1} V'_1. \end{aligned} \tag{2.21}$$

Before substituting in (2.17) and (2.18), I can exploit the nature of the model's MA. In particular, I can use the fact that

$$C_j = (C_1 + M)C_{j-1}, \tag{2.22}$$

which follows from the formula (2.5) and Proposition 1. That is,

$$\begin{aligned} C_j - (C_1 + M)C_{j-1} &= (B_j + B_{j-1}C_1 + B_{j-2}C_2 + \ldots + B_1 C_{j-1}) \\ &\qquad - C_1 C_{j-1} - M(B_{j-1} + B_{j-2}C_1 + \ldots + B_1 C_{j-2}) \\ &= B_1 C_{j-1} - C_1 C_{j-1} \\ &= 0. \end{aligned}$$

Thus, I can write V_0 as follows:

$$V_0 = \Omega + C_1 \Omega C'_1 + (C_1 + M)C_1 \Omega C'_1 (C_1 + M)' + (C_1 + M)^2 C_1 \Omega C'_1 (C_1 + M)^{2'} + \ldots$$

which implies that

$$V_0 = (C_1 + M)V_0(C_1 + M)' + \Omega + C_1 \Omega C'_1 - (C_1 + M)\Omega(C_1 + M)'. \tag{2.23}$$

For V_1,

$$\begin{aligned} V_1 &= C_1 \Omega + (C_1 + M)C_1 \Omega C'_1 + (C_1 + M)^2 \Omega (C_1 + M) + \ldots \\ &= (C_1 + M)V_0 - M\Omega. \end{aligned} \tag{2.24}$$

Substituting (2.23) and (2.24) into (2.21) yields

$$\begin{aligned} V_0 - V_1 V_0^{-1} V'_1 &= V_0 - [(C_1 + M)V_0 - M\Omega]V_0^{-1}[(C_1 + M)V_0 - M\Omega]' \\ &= V_0 - (C_1 + M)V_0(C_1 + M)' + M\Omega(C_1 + M)' \\ &\qquad + (C_1 + M)\Omega M' - M\Omega V_0^{-1}\Omega M' \\ &= \Omega + C_1 \Omega C'_1 - (C_1 + M)\Omega(C_1 + M)' + M\Omega(C_1 + M)' \\ &\qquad + (C_1 + M)\Omega M' - M\Omega V_0^{-1}\Omega M' \\ &= \Omega + M\Omega M' - M\Omega V_0^{-1}\Omega M' \end{aligned}$$

which is the same as (2.20). This proves the first part of the proposition.

For the second part, I need to construct the matrix $\bar{C}_{ols}$ using the relation between the AR coefficients and the MA coefficients in (2.5). In this case,

$$\bar{C}_{ols} = (I - B_{ols})^{-1}.$$

Thus, I have

$$\begin{aligned}
\bar{C}_{ols} &= (I - V_1 V_0^{-1})^{-1} \\
&= (I - [(C_1 + M)V_0 - M\Omega]V_0^{-1})^{-1} \\
&= (I - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (I - (I - M)^{-1}C_1 + (I - M)^{-1}C_1 - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (\bar{C}^{-1} + (I + M + M^2 + \ldots)C_1 - C_1 - M + M\Omega V_0^{-1})^{-1} \\
&= (\bar{C}^{-1} + M(I - M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1})^{-1}. \qquad (2.25)
\end{aligned}$$

The term $M(I-M)^{-1}C_1 + M(\Omega - V_0)V_0^{-1}$ is not generically zero. This proves the second part of the proposition. ■

The term $M\Omega M' - M\Omega V_0^{-1}\Omega M'$ is zero if the RBC model's VAR representation has only one lag (e.g., $M = 0$). It is close to zero if one of the shocks is close to 0. In this latter case, $\Omega V_0^{-1}\Omega \approx \Omega$ and the SVAR user detects correctly the variance of the one shock driving the system. This is true even if the VAR coefficients are wrong (e.g., M is very different than 0).

What happens if we have a VAR with n lags? In this case, the formula is messy but $Ev_t v_t'$ can be written

$$Ev_t v_t' = V_0 - [\, V_1 \quad V_2 \quad \cdots \quad V_n \,] \begin{bmatrix} V_0 & V_1 & \cdots & V_{n-1} \\ V_1' & V_0 & \cdots & V_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ V_{n-1}' & V_{n-2}' & \cdots & V_0 \end{bmatrix}^{-1} \begin{bmatrix} V_1' \\ V_2' \\ \vdots \\ V_n' \end{bmatrix}$$

with $V_j = (C_1 + M)^{j-1}V_1$, where V_0 is the matrix in (2.23) and V_1 is the matrix in (2.24).

2.8. The Propositions for Two Special Cases

In this section, I consider two special cases. The first has $\theta = 0$. The second has $\sigma_\tau = 0$. I show in these very special cases that the SVAR can uncover the true impulse response for hours in response to a technology shock even if only one lag is used in the VAR regression.

2.8.1. Proposition 3a: No capital in the model

Proposition 3a. Assume that θ is set to 0 in the RBC model. If a regression is run of the form

$$X_t = B_{ols} X_{t-1} + v_t$$

with X_t from the RBC model, then the Blanchard-Quah procedure recovers the true impulse response function for hours in response to technology, namely

$$A_j(2,1) = 0 \tag{2.26}$$

for all j.

Proof of Proposition 3a. It is important to note that C_1 is singular in this case. Thus, I can't write M as $C_2 C_1^{-1} - C_1$, but rather, I simply work with:

$$M = \begin{bmatrix} 0 & 0 \\ M(2,1) & \alpha \end{bmatrix}$$

for arbitrary $M(2,1)$ (which is what I would have in the limit as θ goes to 0). It is easy to show that $B_2 = C_2 - C_1^2 = MB_1$, $B_3 = C_3 - B_2 C_2 - B_1 C_1 = MB_2$ and so on. To prove the result in (2.26), I need to show that the errors that a SVAR user encounters in estimating $Ev_t v_t'$ and $\bar{C}$ do not affect the (2,1) elements of the A_j's. Starting with $Ev_t v_t'$:

$$\begin{aligned} Ev_t v_t' &= \Omega + M\Omega M' - M\Omega V_0^{-1} M' \\ &= \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & \sigma_\tau^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \end{aligned}$$

where the second matrix is $M\Omega M' - M\Omega V_0^{-1} M'$ with a nonzero (2,2) element x. The value of x does not affect the result, so I don't need to specify it precisely. Next, I consider the $\bar{C}$ matrix:

$$\begin{aligned} \bar{C}_{ols}^{-1} &= (\bar{C}^{-1} + M(I-M)^{-1} C_1 + M(\Omega - V_0) V_0^{-1})^{-1} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha)/(1-\rho) \end{bmatrix}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right)^{-1} \end{aligned}$$

where y is a nonzero term in the SVAR error. The magnitude of y does not affect the result. Notice that the (1,1) and (1,2) elements of $\bar{C}$ are correctly computed. Notice also that this implies $f = -\bar{C}(1,2)/\bar{C}(1,1) = 0$ and therefore $A_0(2,1) = 0$. For all higher terms, $A_j(2,1) = 0$ because $V_1 V_0^{-1}$ has zeros in the first column. ■

2.8.2. Proposition 3b: Only one shock

Proposition 3b. Assume that $\sigma_\tau = 0$ in the RBC model. If a regression is run of the form

$$X_t = B_{ols} X_{t-1} + v_t$$

with X_t from the RBC model, then the Blanchard-Quah procedure recovers the true impulse responses to technology, namely

$$A_j = D_j Q$$

for all j.

Proof of Proposition 3b. The first part of the proof is concerned with the impact coefficient A_0. I show that $E v_t v_t' = \Omega$ if $\sigma_\tau = 0$, where Ω is the true variance-covariance matrix for the model. This is the main step in showing that the impact coefficient is correct. Then I show that the other coefficients can also be recovered by the SVAR. From Proposition 2, the following holds for the one-lag regression regardless of the size of the shocks:

$$E v_t v_t' = \Omega + M \Omega M' - M \Omega V_0^{-1} \Omega M'.$$

I now show that $\Omega = \Omega V_0^{-1} \Omega$ if $\sigma_\tau = 0$, and therefore $E v_t v_t' = \Omega$. I do this in three steps. First, I show that

$$\Omega = \Omega(1,1) \begin{bmatrix} 1 & \zeta \\ \zeta & \zeta^2 \end{bmatrix} \tag{2.27}$$

where $\zeta = -a/(1 - \theta + \theta a)$. Second, I show that

$$\frac{1}{1 + \zeta \nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta \nu \end{bmatrix} V_0 = \Omega \tag{2.28}$$

where $\nu = -\theta(1 - \gamma_k)(1 - a)/[(\alpha - \gamma_k)a]$. Third, I show that (2.27) and (2.28) imply:

$$\Omega - \Omega V_0^{-1} \Omega = 0. \tag{2.29}$$

Writing out Ω yields

$$
\begin{aligned}
\Omega &= D_0 Q Q' D_0' \\
&= \begin{bmatrix} 1-\theta+\theta a & -\theta b \\ -a & b \end{bmatrix} \begin{bmatrix} \sigma_z^2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1-\theta+\theta a & -a \\ -\theta b & b \end{bmatrix} \\
&= \begin{bmatrix} (1-\theta+\theta a)^2 & -(1-\theta+\theta a)a \\ -(1-\theta+\theta a)a & a^2 \end{bmatrix} \sigma_z^2 \\
&= \Omega(1,1) \begin{bmatrix} 1 & -a/(1-\theta+\theta a) \\ -a/(1-\theta+\theta a) & a^2/(1-\theta+\theta a)^2 \end{bmatrix}
\end{aligned}
$$

and (2.27) holds. Writing out V_0 in (2.28) yields

$$
\frac{1}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (\Omega + C_1 \Omega C_1' + C_2 \Omega C_2' + C_3 \Omega C_3' \ldots). \tag{2.30}
$$

The second term on the right hand side of (2.30) is equal to a 2×2 matrix of zeros:

$$
\begin{aligned}
&\begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} C_1 \Omega C_1' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \begin{bmatrix} \theta^2(1-\gamma_k)^2(1-a)^2 & \theta(1-\gamma_k)(1-a)(\alpha-\gamma_k)a \\ \theta(1-\gamma_k)(1-a)(\alpha-\gamma_k)a & (\alpha-\gamma_k)^2 a^2 \end{bmatrix} \sigma_z^2 \\
&\quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

All higher terms are also equal to 2×2 matrices of zeros:

$$
\begin{aligned}
&\begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} C_j \Omega C_j' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (C_{j-1} D_0 P D_0^{-1} + D_j D_0^{-1})(D_0 Q Q' D_0')(C_{j-1} D_0 P D_0^{-1} + D_j D_0^{-1})' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (\gamma_k^{j-2} D_2)(QQ')(\gamma_k^{j-2} D_2)' \\
&\quad = \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \gamma_k^{2j-4} \begin{bmatrix} [\gamma_k(1-a)\theta(1-\gamma_k)]^2 & \gamma_k^2(1-a)\theta(1-\gamma_k)a(\alpha-\gamma_k) \\ \gamma_k^2(1-a)\theta(1-\gamma_k)a(\alpha-\gamma_k) & [\gamma_k a(\alpha-\gamma_k)]^2 \end{bmatrix} \sigma_z^2 \\
&\quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} (\Omega + C_1 \Omega C_1' + C_2 \Omega C_2' + C_3 \Omega C_3' \ldots) \\
&\qquad = \frac{1}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \Omega \\
&\qquad = \frac{\Omega(1,1)}{1+\zeta\nu} \begin{bmatrix} 1 & \nu \\ \zeta & \zeta\nu \end{bmatrix} \begin{bmatrix} 1 & \zeta \\ \zeta & \zeta^2 \end{bmatrix} \\
&\qquad = \Omega
\end{aligned}
$$

which proves (2.28). Now, I am ready for

$$\begin{aligned}\Omega - \Omega V_0^{-1}\Omega &= \Omega(1,1)\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} - \frac{1}{1+\zeta\nu}\begin{bmatrix}1 & \nu \\ \zeta & \zeta\nu\end{bmatrix}\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} \\ &= \Omega(1,1)\begin{bmatrix}1 & \zeta \\ \zeta & \zeta^2\end{bmatrix} - \frac{1}{1+\zeta\nu}\begin{bmatrix}1+\zeta\nu & \zeta(1+\zeta\nu) \\ \zeta(1+\zeta\nu) & \zeta^2(1+\zeta\nu)\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}\end{aligned}$$

which proves that there is no error in computing $Ev_t v_t'$, that is $Ev_t v_t' = \Omega$. The next step is to show that this is all that is needed for the correct inference. Recall the formulas for the elements of A_0 and λ in Section 3.6. Because $\Omega_{ols} = \Omega$ and det(Ω)=0, it must be the case that $\lambda = 0$. Thus, $A_0(2,1)$ found by the SVAR is

$$A_0 = \begin{bmatrix}\sqrt{\omega_{11}} & 0 \\ \sqrt{\omega_{22}} & 0\end{bmatrix}$$

where $\sqrt{\omega_{jj}} = \sqrt{\Omega(j,j)}$. Using the formulas above, we have $A_0(1,1) = (1-\theta+\theta a)\sigma_z$ and $A_0(2,1) = -a\sigma_z$. Thus, $A_0 = D_0 Q$. This proves that there is no mistaken inference for the impact coefficient. Next, I check A_j for $j > 1$, which is equal to $B_{ols}^j A_0$. For $j = 1$,

$$\begin{aligned}B_{ols}A_0 &= V_1 V_0^{-1} A_0 \\ &= V_1 V_0^{-1} D_0 Q \\ &= (C_1 + M - M\Omega V_0^{-1}) D_0 Q \\ &= C_1 D_0 Q \\ &= (D_0 P + D_1) D_0^{-1} D_0 Q \\ &= D_1 Q\end{aligned}$$

where I have used the fact that $(I - \Omega V_0^{-1}) D_0 Q = 0$. For $j = 2$,

$$\begin{aligned}B_{ols}^2 A_0 &= (V_1 V_0^{-1})^2 A_0 \\ &= (V_1 V_0^{-1}) C_1 D_0 Q \\ &= (C_1 + M - M\Omega V_0^{-1}) C_1 D_0 Q \\ &= C_2 D_0 Q - M\Omega V_0^{-1} C_1 D_0 Q \\ &= C_2 D_0 Q \\ &= (D_0 P^2 + D_1 P + D_2) D_0^{-1} D_0 Q \\ &= D_2 Q\end{aligned}$$

where I have used the fact that $\Omega V_0^{-1} C_1 D_0 Q = 0$. Similarly, I can prove it for higher terms by noting that if $B_{ols}^{j-1} A_0 = C_{j-1} D_0 Q$ holds, then $B_{ols}^{j} A_0 = C_j D_0 Q$ holds and so does the following:

$$\begin{aligned}
B_{ols}^{j} A_0 &= (V_1 V_0^{-1}) C_{j-1} D_0 Q \\
&= (C_1 + M - M \Omega V_0^{-1}) C_{j-1} D_0 Q \\
&= C_j D_0 Q - M \Omega V_0^{-1} C_{j-1} D_0 Q \\
&= C_j D_0 Q \\
&= (D_0 P^j + D_1 P + \ldots + D_j) D_0^{-1} D_0 Q \\
&= D_j Q.
\end{aligned}$$

This establishes that in the case with $\sigma_\tau = 0$, the SVAR uncovers the true impulse responses to technology. ■

What is interesting about the last two propositions is that the special cases are not relevant for modern business cycle theorists. Modern business cycle theorists assume that both capital accumulation and shocks in addition to technology (e.g., distortions to labor) are quantitatively important. Furthermore, adding these factors is not a recent phenomena. They are central to the work following Kydland and Prescott (1982) (which includes my thesis).

3. VAR versus VARMA

In this section, I show that VARMA estimation of the business cycle model also has problems if the eigenvalues of M are close to 1.

In "A Critique of Structural VARs Using Business Cycle Theory," we established that the VAR associated with a standard real business cycle model is infinite,

$$X_t = B_1 X_t + B_2 X_{t-2} + \ldots + v_t, \quad E v_t v_t' = \Omega \tag{3.1}$$

where the elements of X_t are (i) the change in detrended log productivity $(\log(y_t/l_t) - \log(y_{t-1}/l_{t-1})$ and (ii) the quasi-difference of hours $(\log l_t - \alpha \log l_{t-1})$. We showed that $B_j = M B_{j-1}$ and therefore (3.1) can be represented as a VARMA process, namely

$$X_t = (B_1 + M) X_t + v_t - M v_{t-1}. \tag{3.2}$$

The eigenvalues of M are ...