Approximating Transition Dynamics with Discrete Choice

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What we do

- Analyze reforms in settings with discrete choice
  - Eg. occupation choice, default, entry, exit
- Key issue: Costly to compute transition paths
  - Eg. settings with many endogenous states, heterogeneity
- Our approach: Extend perturbation methods
  - fast, scalable and efficient
  - a good initial guess for global methods

Key Insights

- Discrete choice $\rightarrow$ jumps in policy functions
  - point of jump is endogenous
- Discrete time + perturbation requires tracking
  - "small" changes conditional on discrete choice
  - "big" changes due to discrete choice
- Use preference shocks
  - smooths out the discrete choice
  - makes the derivatives used in perturbations well-defined

Plan

- Motivating Example
  - Aiyagari with occupational choice: paid- or self-employment
- Representation of economies with discrete choice
  - more general
- Approximation to transition path
  - using directional derivatives
- Application
  - business taxation

Motivating Example: Aiyagari with Occupational Choice

- Occupational choice in a simple incomplete markets model
  - worker or business owners
- Workers: standard Aiyagari
  - labor productivity $\theta^w_{i,t}$
- Entrepreneurial sector
  - idiosyncratic risk $\theta^b_{i,t}$
  - limited span of control+ financing frictions
  - no ability to issue equity+collateral constraints
- Study taxation of entrepreneurs
  - permanent change in profit tax

Aiyagari with Occupational Choice: Environment

- Workers

$$v_t^w(\theta_{i,t}^w, \theta_{i,t}^b, a_{i,t-1}) = \max_{c_{i,t}, a_{i,t} \geq 0} U(c_{i,t}) + \beta \mathbb{E}_t \left[v_t \left(\theta_{i,t+1}^w, \theta_{i,t+1}^b, \eta_{t+1}, a_{i,t} \right) \right]$$

$$c_{i,t} + a_{i,t} = (1 + R_t - \delta) a_{i,t-1} + (1 - \tau^w) W_t \theta_{i,t}^w + T_t$$

- Business owners

$$v_t^b(\theta_{i,t}^w, \theta_{i,t}^b, a_{i,t-1}) = \max_{c_{i,t}, a_{i,t} \geq 0} U(c_{i,t}) + \beta \mathbb{E}_t \left[v_t \left(\theta_{i,t+1}^w, \theta_{i,t+1}^b, \eta_{t+1}, a_{i,t} \right) \right]$$

$$c_{i,t} + a_{i,t} = (1 + R_t - \delta) a_{i,t-1} + (1 - \tau^b) \pi_{i,t} + T_t$$

$$\pi_{i,t} = \max_{n_{i,t} k_{i,t} \leq \chi a_{i,t-1},} \theta_{i,t}^b k_{i,t}^\alpha n_{i,t}^\nu - R_t k_{i,t} - W_t n_{i,t}$$

- Continuation value $v_t = \max_{d_{i,t} \in \{0,1\}} d_{i,t} \left(v_t^w + \eta_{i,t} \right) + (1 - d_{i,t}) v_t^b$
    - $\eta_{i,t}$ are iid taste shocks $\sim \Gamma$

Aiyagari with Occupational Choice: Environment

- Corporate sector

$$\max_{N_t^c, K_t^c} \Theta \left(K_t^c\right)^{\alpha} \left(N_t^c\right)^{1-\alpha} - W_t N_t^c - R_t K_t^c$$

- Market clearing

$$K_t^c + \int k_{i,t} di = \int a_{i,t-1} di,$$

$$N_t^c + \int (1 - d_{i,t}) n_{i,t} di = \int d_{i,t} \theta_{i,t}^w di$$

$$T_t = \int \tau^b \pi_{i,t} (1 - d_{i,t}) di + \int \tau^w W_t d_{i,t} \theta_{i,t}^w di$$

- Exogenous stochastic processes: $\theta_{i,t} = [\log \theta_{i,t}^b, \log \theta_{i,t}^w]$

$$\theta_{i,t} = \rho_\theta \theta_{i,t-1} + \varepsilon_{i,t}^\theta$$

Aiyagari with Occupational Choice: Tax Reform

- Assume economy is at long run steady state for some $\tau = (\tau^w, \tau^b)$
- Consider a policy change $\tau \to \tau'$
  - What are the transition dynamics?
- Compute optimal fiscal policy
  - Use the transition to assign welfare gains to each reform

Notation

- Need notation to store endogenous objects and represent optimality
- Individual choices
  - states
  - choices given occupation
  - indicator for occupation
- Aggregates
  - measures over endogenous states
  - endogenous variables that are not measures
  - states

Notation

- Need notation to store endogenous objects and represent optimality
- Individual
  - states: $(a_{i,t-1}, \theta_{i,t})$
  - choices given occupation: $x_{i,t}^{d}$ for $d \in \{w, b\}$
    - $a_{i,t}^{d} \in x_{i,t}^{d}$: predetermined
    - $v_{i,t}^{d} \in x_{i,t}^{d}$: value functions
  - indicator for occupation: $d_{i,t} = \iota\left(v_{i,t}^{b} - v_{i,t}^{w} \leq \eta_{i,t}\right)$
    - $x_{i,t} = d_{i,t} x_{i,t}^{w} + (1 - d_{i,t}) x_{i,t}^{b}$
- Aggregates
  - measures (CDF) over individual states: Ω_t
  - endogenous variables that are not measures: X_t, Y_t, A_t
    - $A_t \in X_t$: predetermined aggregates
    - $Y_t = [A_{t-1}, X_t, X_{t+1}]^{\mathrm{T}}$: agg. variables relevant for period t
  - states: $Z_t = [A_{t-1}, \Omega_t]$

Representation without Discrete Choice

- Optimality conditions of agents with idiosyncratic shocks:

$$F\left(\theta_{i,t}, a_{i,t-1}, x_{i,t}, \mathbb{E}_{i,t} x_{i,t+1}, Y_t\right) = 0 \text{ for all } i, t$$

- All other equilibrium conditions:

$$G\left(\int x_{i,t} di, Y_t\right) = 0 \text{ for all } t$$

- Equilibrium Paths: $\left\{X_t, x_t\left(\theta^t\right)\right\}_{t,\theta^t}$ that satisfies equations F and G given initial conditions $\left(\left\{a_{i,-1}, \theta_{i,0}\right\}_i, A_{-1}\right)$

Representation with Discrete Choice

- Optimality conditions of agents with idiosyncratic shocks:

$$F^d\left(\theta_{i,t}, a_{i,t-1}, x^d_{i,t}, \mathbb{E}_{i,t}x_{i,t+1}, Y_t\right) = 0 \text{ for all } i, t, d \in \{w, b\}$$

  - discrete choice threshold $\kappa_{i,t} \equiv v^b_{i,t} - v^w_{i,t}$

- All other equilibrium conditions:

$$G\left(\int x_{i,t} di, Y_t\right) = 0 \text{ for all } t$$

- Equilibrium Paths: $\left\{X_t, x^d_t\left(\theta^t, \eta^t\right), \kappa_t\left(\theta^t, \eta^t\right)\right\}_{t,\theta^t,\eta^t}$ that satisfies equations F and G given initial conditions $\left(\{a_{i,-1}, \theta_{i,0}\}_i, A_{-1}\right)$

Approximations to Equilibrium Paths

- Two steady states
  - pre-reform $Z_{-1} = [A_{-1}, \Omega_{-1}]^{\mathrm{T}}$
  - post-reform $Z^* = [A^*, \Omega^*]^{\mathrm{T}}$
- Transition path approximated by perturbation in Z
  - approximate around post-reform SS
  - scale deviations of initial state: $Z_0 = Z^* + \sigma (Z_{-1} - Z^*)$
  - equilibrium path $X_t(\sigma)$
    - $\sigma = 0 \implies$ post-reform steady state
    - $\sigma = 1 \implies$ transition from pre-reform steady state
- Use Taylor expansions w.r.t. σ to find various orders of approximations
  - use the recursive (state-space) representation of equilibrium

Using the Recursive Representation

- Let $Z = [A, \Omega]^T$ be aggregate states and (a, θ, Z) individual states
  - $\overline{x}(a, \theta, \eta, Z), \overline{x}^d(a, \theta, Z)$, $\overline{X}(Z)$ are indiv + agg policy functions
- Recursive representation:
  - mappings F^d, G as before but serve as functional equations
  - $\overline{\Omega}(Z)[a', \theta']$ next period CDF

$$\iiint \iota\left(\overline{a}(a, \theta, \eta, Z) \leq a'\right) \iota(\rho_\theta \theta + \varepsilon \leq \theta')\mu(\varepsilon)\, d\varepsilon d\Gamma(\eta) d\Omega(a, \theta)$$

- Approximate X_t using directional derivatives of policy functions
  - derivatives (Frechet) w.r.t. Z: $\overline{X}_Z$, $\overline{X}_Z$, $\overline{x}_Z^d(a, \theta)$, $\overline{Z}_Z$, ...
  - directional derivative in direction $\hat{Z}$: $\hat{X} \equiv \overline{X}_Z \cdot \hat{Z}$
  - pick appropriate directions to implement the transition path

Directions for Transition Path: 1st order

- To the 1^{st} order:

$$X_t = \overline{X} + \hat{X}_t + O\left(\|Z_{-1} - Z^*\|^2\right),$$

where directions $\left\{\hat{Z}_t\right\}$

$$\hat{Z}_0 := Z_{-1} - Z^*, \quad \hat{Z}_t := \overline{Z}_Z \cdot \hat{Z}_{t-1},$$

$$\hat{X}_t := \overline{X}_Z \cdot \hat{Z}_t.$$

- Economic intuition:
  - $\{\hat{Z}_t\}_t$ traces out LoM for Ω following a change in Z
  - $\left\{\hat{X}_t\right\}_t$ is the impulse response to that change

Characterizing $\left\{\hat{X}_t\right\}_t$

- Goal: To derive a linear system for $\left\{\hat{X}_t\right\}$
  - $\left\{\hat{X}_t\right\}$ satisfy $G_Z\left(\int \overline{x} d\Gamma d\Omega, \overline{Y}\right) \cdot \hat{Z}_t = 0$
- Need to track individual behavior and the measure
  - $\hat{x}_t \equiv \overline{x}_Z \cdot \hat{Z}_t, \quad \hat{\Omega}_t \equiv \overline{\Omega}_Z \cdot \hat{Z}_{t-1}$
- How does discrete choice manifest?
  - individual policies have jumps
  - the point of jump is endogenous
- Perturbing state Z changes $\left(\hat{x}, \hat{\Omega}\right)$
  - "small" changes conditional on occupation
  - "big" changes due to change in occupation

Derive a Linear System for $\left\{\hat{X}_t\right\}$

- First derivative of G in direction $\hat{Z}_t$:

$$\mathrm{G}_Y \hat{Y}_t + \mathrm{G}_x \left(\iint \overline{x} d\Gamma d\Omega\right)_Z \cdot \hat{Z}_t = 0,$$

where

$$\hat{Y}_t = \left[\mathrm{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

and recall $\hat{Z}_t = [\hat{A}_t, \hat{\Omega}_t]$

$$\left(\iint \overline{x} d\Gamma d\Omega\right)_Z \cdot \hat{Z}_t = \left(\int \overline{x} d\Gamma d\Omega^*\right)_Z \cdot \hat{Z}_t + \iint \overline{x} d\Gamma d\hat{\Omega}_t$$

How does discrete choice manifest?

- Consider tracking a change in the aggregate: $\iint \overline{x} d\Gamma d\Omega^*$

$$\iint \overline{x} d\Gamma d\Omega^* = \int \int_{-\infty}^{\overline{\kappa}(a,\theta,Z)} \overline{x}^w(a,\theta,Z) d\Gamma(\eta) d\Omega^*(a,\theta) + \int \int_{\overline{\kappa}(a,\theta,Z)}^{\infty} \overline{x}^b(a,\theta,Z) d\Gamma(\eta) d\Omega^*(a,\theta)$$

- Differentiating in direction $\hat{Z}_t$

$$\left(\int \overline{x} d\Gamma d\Omega^* \right)_Z \cdot \hat{Z}_t = \int \underbrace{\overline{\Gamma}^w \bar{x}_Z^w \cdot \hat{Z}_t + \overline{\Gamma}^b \bar{x}_Z^b \cdot \hat{Z}_t}_{\hat{x}_t} d\Omega^* + \int \left(\overline{x}^w - \overline{x}^b \right) \Gamma' \left(\bar{\kappa} \right) \underbrace{\bar{\kappa}_Z \cdot \hat{Z}_t}_{\hat{\kappa}_t} d\Omega^*$$

where $\overline{\Gamma}^w(a,\theta) = \Gamma(\overline{\kappa}(a,\theta))$ and $\overline{\Gamma}^b(a,\theta) = 1 - \Gamma(\overline{\kappa}(a,\theta))$ and $\hat{\kappa}_t = \hat{v}_t^b - \hat{v}_t^w$ with $v_t^d \in \hat{x}_t^d$

- Contains both both
  - “small” changes conditional on occupation
  - “big” changes due to change in occupation

How does discrete choice manifest?

- Differentiating

$$\iint \overline{x} d\Gamma d\Omega^* = \int \int_{-\infty}^{\overline{\kappa}(a,\theta,Z)} \overline{x}^w(a,\theta,Z) d\Gamma(\epsilon) d\Omega^*(a,\theta) + \int \int_{\overline{\kappa}(a,\theta,Z)}^{\infty} \overline{x}^b(a,\theta,Z) d\Gamma(\epsilon) d\Omega^*(a,\theta)$$

  implies

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implies

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where $\overline{\Gamma}^w(a,\theta) = \Gamma(\overline{\kappa}(a,\theta))$ and $\overline{\Gamma}^b(a,\theta) = 1 - \Gamma(\overline{\kappa}(a,\theta))$ and $\hat{\kappa}_t = \hat{v}_t^b - \hat{v}_t^w$ with $v_t^d \in \hat{x}_t^d$

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where $\overline{\Gamma}^w(a,\theta) = \Gamma(\overline{\kappa}(a,\theta))$ and $\overline{\Gamma}^b(a,\theta) = 1 - \Gamma(\overline{\kappa}(a,\theta))$ and $\hat{\kappa}_t = \hat{v}_t^b - \hat{v}_t^w$ with $v_t^d \in \hat{x}_t^d$

- Contains both
  - "small" changes conditional on occupation
  - "big" changes due to change in occupation

Derive a Linear System for $\left\{\hat{X}_t\right\}$

- First derivative of G in direction $\hat{Z}_t$:

$$\mathsf{G}_Y \hat{Y}_t + \mathsf{G}_x \left(\iint \overline{x} d\Gamma d\Omega\right)_Z \cdot \hat{Z}_t = 0,$$

where

$$\hat{Y}_t = \left[\mathsf{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

and recall $\hat{Z}_t = [\hat{A}_t, \hat{\Omega}_t]$

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- We already know $d\Omega^*$, G_x, G_Y, P
- If we can express $\hat{x}_t^d$ and $\hat{\Omega}_t$ in terms of $\left\{\hat{X}_s\right\}_s$, we found a way to solve 1^{st} order

Differentiate F^d for individual changes $\left\{\hat{x}_t^d\right\}$

- First derivative of F^d in direction $\hat{Z}_t$:

$$\hat{x}_t^d(a,\theta) = \sum_{s=0}^{\infty} \mathrm{x}_s^d(a,\theta)\,\hat{Y}_{t+s}$$

$$\mathrm{x}_0^d(a,\theta) = -\left(\mathrm{F}_x^d(a,\theta) + \mathrm{F}_{x^e}^d(a,\theta)\,\mathbb{E}\left[\overline{\mathrm{x}}_a|a,\theta\right]\mathrm{p}\right)^{-1}\mathrm{F}_Y^d(a,\theta)\,,$$
$$\mathrm{x}_{s+1}^d(a,\theta) = -\left(\mathrm{F}_x^d(a,\theta) + \mathrm{F}_{x^e}^d(a,\theta)\,\mathbb{E}\left[\overline{\mathrm{x}}_a|a,\theta\right]\mathrm{p}\right)^{-1}\mathrm{F}_{x^e}^d(a,\theta)\,\mathbb{E}\left[\mathrm{x}_s|a,\theta\right]$$

- RHS already known $\Longrightarrow$ easy and fast way to compute $\{\mathrm{x}_s\}_s$
  - No numerical differentiation!
- Discrete choice: Immediately gives us change in threshold
  - $\hat{\kappa}_t = \hat{v}_t^b - \hat{v}_t^w$ with $v_t^d \in \hat{x}_t^d$

Differentiate $\overline{\Omega}$ for changes in measure $\left\{\hat{\Omega}_t\right\}$

- Perturbation can effect distribution through
  - "small" changes in the state conditional on occupational choice
  - "big" changes in the state due to changes in occupation
- Derive a recursive law for $\frac{d}{da}\frac{d}{d\theta}\hat{\Omega}_t$ using two sets of operators
  - Time shift: Push changes in distribution $t \implies$changes in distribution $t+1$
    - for "small" changes: $\mathcal{L}^{(a)}$
    - for "big" changes: $\mathcal{L}$
  - Policy shift: Push changes in policies $t \implies$changes in distribution $t+1$
    - for "small" changes: $\mathcal{M}^{(a),d}$
    - for "big" changes: $\mathcal{M}$
- Aggregate $\int \overline{x} d\hat{\Omega}_t$ using operators $\mathcal{I}$ and $\mathcal{I}^{(a)}$

Differentiate $\overline{\Omega}$ for changes in measure $\left\{\hat{\Omega}_t\right\}$

- Change in measure Ω

$$\frac{d}{da}\frac{d}{d\theta}\hat{\Omega}_t = \underbrace{-\frac{d}{da}\hat{\omega}_t^{(a)}}_{\text{``small'' changes}} + \underbrace{\hat{\omega}_t}_{\text{``big'' changes}}$$

$$\hat{\omega}_{t+1}^{(a)} = \mathcal{L}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{M}^{(a),w} \cdot \hat{a}_t^w + \mathcal{M}^{(a),b} \cdot \hat{a}_t^b$$

$$\hat{\omega}_{t+1} = \Lambda \cdot \hat{\omega}_t + \mathcal{L} \cdot \hat{\omega}_t^{(a)} + \mathcal{M} \cdot \hat{\kappa}_t$$

- Aggregation

$$\int \overline{x} d\hat{\Omega}_t = \mathcal{I}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{I} \cdot \hat{\omega}_t$$

Time and Policy Shifts for "Small" Changes

$$\hat{\omega}^{(a)}_{t+1} = \mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)}_t + \mathcal{M}^{(a),w} \cdot \hat{a}^w_t + \mathcal{M}^{(a),b} \cdot \hat{a}^b_t$$

- Policy shifts: policies $t \implies$ distribution $t+1$

$$\mathcal{M}^{(a),d} \cdot \hat{a}^d \langle a', \theta' \rangle = \iint \bar{\Lambda}^d(a', \theta', a, \theta) \bar{\Gamma}^d(a, \theta) \omega^*(a, \theta) \hat{a}^d(a, \theta) da d\theta$$

- Time shifts: distribution $t \implies$distribution $t+1$

$$\mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)} \langle a', \theta' \rangle = \iint \sum_d \bar{\Lambda}^d(a', \theta', a, \theta) \bar{\Gamma}^d(a, \theta) \bar{a}^d_a(a, \theta) \hat{\omega}^{(a)}(a, \theta) da d\theta$$

Time and Policy Shifts for "Small" Changes

$$\hat{\omega}^{(a)}_{t+1} = \mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)}_t + \mathcal{M}^{(a),w} \cdot \hat{a}^w_t + \mathcal{M}^{(a),b} \cdot \hat{a}^b_t$$

- Policy shifts: policies $t \implies$ distribution $t+1$

$$\mathcal{M}^{(a),d} \cdot \hat{a}^d \langle a', \theta' \rangle = \iint \underbrace{\bar{\Lambda}^d(a', \theta', a, \theta)\bar{\Gamma}^d(a, \theta)}_{\substack{\text{average with SS} \\ \text{transition density}}} \quad \underbrace{\omega^*(a, \theta)}_{\substack{\text{weight by SS} \\ \text{density}}} \quad \underbrace{\hat{a}^d(a, \theta)}_{\substack{\text{small change} \\ \text{in state}}} \quad da d\theta$$

- Time shifts: distribution $t \implies$distribution $t+1$

$$\mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)} \langle a', \theta' \rangle = \iint \sum_d \bar{\Lambda}^d(a', \theta', a, \theta)\bar{\Gamma}^d(a, \theta)\bar{a}^d_a(a, \theta)\hat{\omega}^{(a)}(a, \theta) da d\theta$$

Time and Policy Shifts for "Small" Changes

$$\hat{\omega}^{(a)}_{t+1} = \mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)}_t + \mathcal{M}^{(a),w} \cdot \hat{a}^w_t + \mathcal{M}^{(a),b} \cdot \hat{a}^b_t$$

- Policy shifts: policies $t \implies$ distribution $t+1$

$$\mathcal{M}^{(a),d} \cdot \hat{a}^d \langle a', \theta' \rangle = \iint \underbrace{\bar{\Lambda}^d(a', \theta', a, \theta)\bar{\Gamma}^d(a, \theta)}_{\text{average with SS transition density}} \quad \underbrace{\omega^*(a, \theta)}_{\text{weight by SS density}} \quad \underbrace{\hat{a}^d(a, \theta)}_{\text{small change in state}} \quad da d\theta$$

- Time shifts: distribution $t \implies$distribution $t+1$

$$\mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)} \langle a', \theta' \rangle = \iint \sum_d \bar{\Lambda}^d(a', \theta', a, \theta)\bar{\Gamma}^d(a, \theta)\bar{a}^d_a(a, \theta)\hat{\omega}^{(a)}(a, \theta) da d\theta$$

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$$\mathcal{M}^{(a),d} \cdot \hat{a}^d \langle a', \theta' \rangle = \iint \bar{\Lambda}^d(a', \theta', a, \theta) \bar{\Gamma}^d(a, \theta) \omega^*(a, \theta) \hat{a}^d(a, \theta) da d\theta$$

- Time shifts: distribution $t \Longrightarrow$ distribution $t+1$

$$\mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)} \langle a', \theta' \rangle = \iint \sum_d \underbrace{\bar{\Lambda}^d(a', \theta', a, \theta) \bar{\Gamma}^d(a, \theta)}_{\text{average with SS transition density}} \quad \underbrace{\bar{a}^d_a(a, \theta)}_{\text{changes next period's state}} \quad \underbrace{\hat{\omega}^{(a)}(a, \theta)}_{\text{small change in initial state}} \quad da d\theta$$

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$$\hat{\omega}^{(a)}_{t+1} = \mathcal{L}^{(a)} \cdot \hat{\omega}^{(a)}_t + \mathcal{M}^{(a),w} \cdot \hat{a}^w_t + \mathcal{M}^{(a),b} \cdot \hat{a}^b_t$$

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$$\mathcal{M}^{(a),d} \cdot \hat{a}^d \langle a', \theta' \rangle = \iint \bar{\Lambda}^d(a', \theta', a, \theta) \bar{\Gamma}^d(a, \theta) \omega^*(a, \theta) \hat{a}^d(a, \theta) da d\theta$$

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$$\hat{\omega}_{t+1}^{(a)} = \mathcal{L}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{M}^{(a),w} \cdot \hat{a}_t^w + \mathcal{M}^{(a),b} \cdot \hat{a}_t^b$$

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Time and Policy Shifts for "Big" Changes

$$\hat{\omega}_{t+1} = \Lambda \cdot \hat{\omega}_t + \mathcal{L} \cdot \hat{\omega}_t^{(a)} + \mathcal{M} \cdot \hat{\kappa}_t$$

- Policy shifts: policies $t \implies$ distribution $t+1$

$$\mathcal{M} \cdot \hat{\kappa}\langle a', \theta' \rangle = \iint \left(\bar{\Lambda}^w(a', \theta', a, \theta) - \bar{\Lambda}^b(a', \theta', a, \theta) \right) \Gamma'(\bar{\kappa}(a, \theta)) \omega^*(a, \theta) \hat{\kappa}(a, \theta) da d\theta$$

- Time shifts: distribution $t \implies$distribution $t+1$

$$\mathcal{L} \cdot \hat{\omega}^{(a)}\langle a', \theta' \rangle = \iint \left(\bar{\Lambda}^w(a', \theta', a, \theta) - \bar{\Lambda}^b(a', \theta', a, \theta) \right) \Gamma'(\bar{\kappa}(a, \theta)) \bar{\kappa}_a(a, \theta) \hat{\omega}^{(a)}(a, \theta) da d\theta$$

$$\Lambda \cdot \hat{\omega}\langle a', \theta' \rangle = \iint \sum_d \bar{\Lambda}^d(a', \theta', a, \theta) \Gamma^d(a, \theta) \hat{\omega}(a, \theta) da d\theta$$

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- Time shifts: distribution $t \Longrightarrow$ distribution $t+1$

$$\mathcal{L} \cdot \hat{\omega}^{(a)}\langle a', \theta' \rangle = \iint \left(\bar{\Lambda}^w(a', \theta', a, \theta) - \bar{\Lambda}^b(a', \theta', a, \theta)\right) \Gamma'(\bar{\kappa}(a, \theta))\bar{\kappa}_a(a, \theta)\hat{\omega}^{(a)}(a, \theta) da d\theta$$

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Aggregation

$$\int \overline{x} d\hat{\Omega}_t = \mathcal{I}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{I} \cdot \hat{\omega}_t$$

- $\mathcal{I}$ and $\mathcal{I}^{(a)}$: distribution $\Longrightarrow$ aggregate choices

$$\begin{aligned} \mathcal{I}^{(a)} \cdot \hat{\omega}^{(a)} = & \iint \sum_d \overline{x}_a^d(a,\theta)\Gamma^d(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta \\ & + \iint \left(\overline{x}^w(a,\theta) - \overline{x}^b(a,\theta)\right)\Gamma'\left(\overline{\kappa}(a,\theta)\right)\overline{\kappa}_a(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta \end{aligned}$$

$$\mathcal{I} \cdot \hat{\omega} = \iint \sum_d \overline{x}^d(a,\theta)\overline{\Gamma}^d(a,\theta)\hat{\omega}(a)dad\theta$$

- Captures both “small changes” and “big changes” terms

Aggregation

$$\int \overline{x} d\hat{\Omega}_t = \mathcal{I}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{I} \cdot \hat{\omega}_t$$

- $\mathcal{I}$ and $\mathcal{I}^{(a)}$: distribution $\Longrightarrow$ aggregate choices

$$\mathcal{I}^{(a)} \cdot \hat{\omega}^{(a)} = \iint \sum_d \overline{x}_a^d(a,\theta)\Gamma^d(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta$$
$$+ \iint \left(\overline{x}^w(a,\theta) - \overline{x}^b(a,\theta)\right) \Gamma'\left(\overline{\kappa}(a,\theta)\right) \overline{\kappa}_a(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta$$

$$\mathcal{I} \cdot \hat{\omega} = \iint \sum_d \overline{x}^d(a,\theta)\overline{\Gamma}^d(a,\theta)\hat{\omega}(a)dad\theta$$

- Captures both "small changes" and "big changes" terms

Aggregation

$$\int \overline{x} d\hat{\Omega}_t = \mathcal{I}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{I} \cdot \hat{\omega}_t$$

- $\mathcal{I}$ and $\mathcal{I}^{(a)}$: distribution $\Longrightarrow$ aggregate choices

$$\mathcal{I}^{(a)} \cdot \hat{\omega}^{(a)} = \iint \sum_d \overline{x}_a^d(a,\theta)\Gamma^d(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta + \iint \left(\overline{x}^w(a,\theta) - \overline{x}^b(a,\theta)\right)\Gamma'\left(\overline{\kappa}(a,\theta)\right)\overline{\kappa}_a(a,\theta)\hat{\omega}^{(a)}(a,\theta)dad\theta$$

$$\mathcal{I} \cdot \hat{\omega} = \iint \sum_d \overline{x}^d(a,\theta)\overline{\Gamma}^d(a,\theta)\hat{\omega}(a)dad\theta$$

- Captures both "small changes" and "big changes" terms

Differentiate $\overline{\Omega}$ for changes in measure $\left\{\hat{\Omega}_t\right\}$

- Change in measure Ω

$$\frac{d}{da}\frac{d}{d\theta}\hat{\Omega}_t = \underbrace{-\frac{d}{da}\hat{\omega}_t^{(a)}}_{\text{“small” changes}} + \underbrace{\hat{\omega}_t}_{\text{“big” changes}}$$

$$\hat{\omega}_{t+1}^{(a)} = \mathcal{L}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{M}^{(a),w} \cdot \hat{a}_t^w + \mathcal{M}^{(a),b} \cdot \hat{a}_t^b$$

$$\hat{\omega}_{t+1} = \Lambda \cdot \hat{\omega}_t + \mathcal{L} \cdot \hat{\omega}_t^{(a)} + \mathcal{M} \cdot \hat{\kappa}_t$$

- Aggregation

$$\int \overline{x} d\hat{\Omega}_t = \mathcal{I}^{(a)} \cdot \hat{\omega}_t^{(a)} + \mathcal{I} \cdot \hat{\omega}_t$$

Putting together

- $\left\{\hat{X}_t\right\}_t$ solves

$$\mathrm{G}_Y \hat{Y}_t + \mathrm{G}_x \sum_{s=0}^{\infty} \mathrm{J}_{t,s} \hat{Y}_s + \mathrm{G}_x \mathrm{J}_t^{TD} = 0 \text{ for all } t$$

$$\hat{Y}_t = \left[\mathrm{P}\hat{X}_{t-1}, \hat{X}_t, \hat{X}_{t+1}\right]^{\mathrm{T}}$$

$$\mathrm{J}_{t,s} = \mathrm{J}_{t-1,s-1} + \sum_d \mathcal{IL}_t \cdot \mathcal{M}^{(a),d} \cdot \mathrm{px}_s^d + \mathcal{I} \cdot \left(\bar{\Lambda}\right)^{t-1} \cdot \mathcal{M} \cdot \left(\mathrm{v}_s^b - \mathrm{v}_s^w\right)$$

$$\mathcal{IL}_t = \mathcal{IL}_{t-1} \cdot \mathcal{L}^{(a)} + \mathcal{I} \cdot \left(\bar{\Lambda}\right)^{t-2} \cdot \mathcal{L}$$

$$\mathrm{J}_t^{TD} = \mathcal{I} \cdot \Lambda^t \cdot \hat{\omega}_0$$

where $\mathrm{J}_{0s} = \int \mathrm{x}_s d\Omega^* + \int (\bar{x}^w - \bar{x}^b)\Gamma'(\bar{\kappa})\kappa_s d\Omega^*$ and $\mathrm{P}\hat{X}_{-1} = \mathrm{P}\left(\overline{X}_{-1} - \overline{X}\right)$, and $\lim_{t\to\infty} \hat{X}_t = 0$.

Numerical implementation

- User inputs: SS policy functions (splines) and equations describing competitive equilibrium
- First-order: need G_x, G_Y, J_t^{TD} and $\{\mathsf{J}_{t,s}\}_{t,s}$ to solve

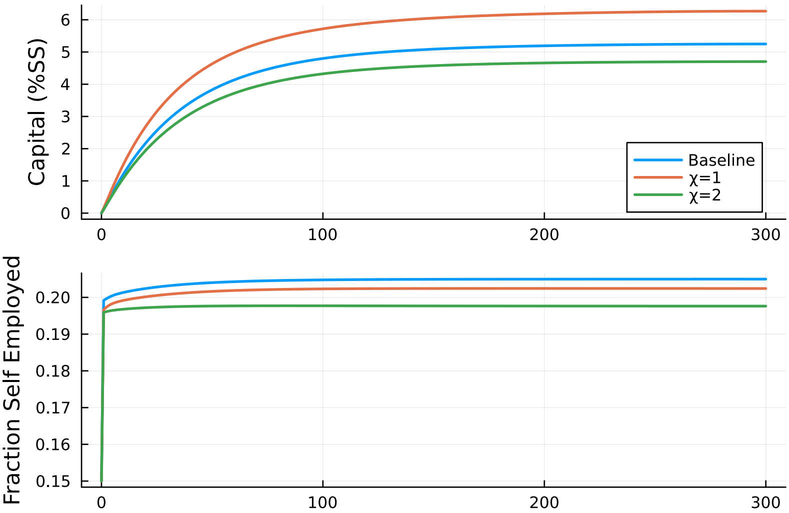
$$\mathsf{G}_Y \hat{Y}_t + \mathsf{G}_x \sum_{s=0}^{\infty} \mathsf{J}_{t,s} \hat{Y}_s + \mathsf{G}_x \mathsf{J}_t^{TD} = 0 \text{ for all } t$$

- G_x, G_Y automatically differentiate G and evaluate at SS
- For $\mathsf{J}_{t,s}$ and J_t^{TD} need
  - Operators $\mathcal{I}^{(a)}, \mathcal{I}$, $\mathcal{L}^{(a)}, \mathcal{L}$ and $\mathcal{M}^{(a)}, \mathcal{M}$: sparse matrices using SS transition matrix
  - Coeffs $\{\mathsf{x}_t\}_t$: recursively using linear algebra with pre-computed basis matrices
- Solve the linear system $\{\hat{X}_t\}_{t=0}^{T}$ by truncation

Illustrative Calibration

- Technology: Standard parameters
- Preferences: $U(c) = \frac{c^{1-\text{ra}}}{1-\text{ra}}$
  - risk aversion ra set to 2
- Shocks
  - $\theta_{i,t}$: Finite-state Markov to match paid- and self-employed earnings
  - $\eta_{i,t}$: Gumbel with $\sigma_{\varepsilon} \approx 0$
- Financial frictions
  - Baseline $\chi = 1.5$ to get a max leverage ratio of $\frac{1}{3}$
  - Robustness with higher and lower values
- Two experiments
  - transition paths for a change in τ_b
  - find optimal τ_b

Transition paths: Decrease τ_b from 30% $\rightarrow$ 10%



Optimal Business Tax

- Welfare: Date $t = 0$ aggregation of values

$$W_0 \equiv \int v_{0,i,t} di$$

- Implementation
  - add W_t in X_t and augment G
- Offsetting economic forces
  - tax returns to capital
    - Redistribution (Utilitarian obj.)
    - Over savings (Aiyagari)
  - subsidize returns to capital
    - Misallocation (Guvenen et al)

Optimal Business Tax

Maximum Leverage	χ	Optimal τ	Welfare Gain (%C)
0%	1.0	-0.21	2.5
33%	1.5	-0.08	1.8
100%	2.0	0.00	1.6

Next Steps

- Second order
  - all steps naturally extend to second order
  - worked out the algebra need to update the code
- Speed and Accuracy
  - compare transition dynamics to global solution
- Optimal tax mix
  - endogenous labor
  - should government rely on business or labor taxes

FOC - Workers

- Optimality conditions for agents with idiosyncratic risk (workers):

$$
\begin{aligned}
c_{i,t} + a_{i,t} - (1 + R_t - \delta)\, a_{i,t-1} - W_t \exp(\theta_{i,t}) - T_t &= 0 \\
v^w_{a,i,t} - (1 + R_t - \delta)\, u_c(c_{i,t}) &= 0 \\
n_{i,t} + \theta^w_{i,t} &= 0 \\
k_{i,t} &= 0 \\
\pi_{i,t} &= 0 \\
v^w_{i,t} - u(c_{i,t}) - \beta \mathbb{E}_t v_{i,t+1} &= 0 \\
u_c(c_{i,t}) + \zeta_{i,t} - \beta \mathbb{E}_t v_{a,i,t+1} &= 0 \\
a_{i,t}\zeta_{i,t} &= 0
\end{aligned}
$$

FOC - Business Owneers

- Optimality conditions for agents with idiosyncratic risk (business owners):

$$
\begin{aligned}
c_{i,t} + a_{i,t} - (1 + R_t - \delta)\, a_{i,t-1} - (1-\tau)\pi_{i,t} - T_t &= 0 \\
v^b_{a,i,t} - (1 + R_t - \delta)\, u_c(c_{i,t}) - \chi\zeta_{i,t} &= 0 \\
v^b_{i,t} - u(c_{i,t}) - \beta\mathbb{E}_t v_{i,t+1} &= 0 \\
u_c(c_{i,t}) + \zeta_{i,t} - \beta\mathbb{E}_t v_{a,i,t+1} &= 0 \\
\pi_{i,t} - \theta^b_{i,t} k^\alpha_{i,t} n^\nu_{i,t} - R_t k_{i,t} - W_t n_{i,t} &= 0 \\
\alpha\theta^b_{i,t} k^{\alpha-1}_{i,t} n^\nu_{it} - R_t - \xi_{i,t} &= 0 \\
\nu\theta^b_{i,t} k^\alpha_{i,t} n^{\nu-1}_{i,t} - W_t &= 0 \\
a_{i,t}\zeta_{i,t} &= 0 \\
(\chi a_{i,t-1} - k_{i,t})\xi_{i,t} &= 0
\end{aligned}
$$

Occupation Choice

- Occupational choice

$$v_{i,t} = \sigma_\varepsilon \log\left(\exp\left(v^w_{i,t}/\sigma_\varepsilon\right) + \exp\left(v^b_{i,t}/\sigma_\varepsilon\right)\right)$$

$$v_{a,i,t} = \frac{1}{1 + \exp\left(\left(v^b_{i,t} - v^w_{i,t}\right)/\sigma_\varepsilon\right)} v^w_{a,i,t} + \frac{1}{1 + \exp\left(\left(v^w_{i,t} - v^b_{i,t}\right)/\sigma_\varepsilon\right)} v^b_{a,i,t}$$

Market Clearing Etc.

- Aggregate Conditions

$$A_t - \int a_{i,t} di = 0$$

$$A_{t-1} - K_t^C - \int k_{i,t} di = 0$$

$$N_t^C + \int n_{i,t} di = 0$$

$$R_t - \alpha \Theta \left(K_t^C / N_t^C \right)^{\alpha - 1} = 0$$

$$W_t - (1 - \alpha) \Theta \left(K_t^C / N_t^C \right)^{\alpha} = 0$$